

PARITY OF POLYNOMIAL MULTIPLIER SEQUENCES FOR THE CHEBYSHEV BASIS

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ABSTRACT. We demonstrate that if $p \in \mathbb{R}[x]$ and p is not an even function, then $\{p(k)\}_{k=0}^{\infty}$ is not a multiplier sequence for the basis of Chebyshev polynomials of the first kind. We also give a characterization of geometric multiplier sequences for the Chebyshev basis.

1. INTRODUCTION

A polynomial is *hyperbolic* if it either has only real zeros or if it is identically zero. An operator K on the vector space of real polynomials $\mathbb{R}[x]$ is called *hyperbolicity preserving* if $K(p)$ is hyperbolic whenever p is hyperbolic. Let $B = \{b_k(x)\}_{k=0}^{\infty}$ be a basis for $\mathbb{R}[x]$. To any sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$, we can associate an operator K on the vector space $\mathbb{R}[x]$ defined by

$$(1) \quad K \left[\sum_{k=0}^n a_k b_k(x) \right] = \sum_{k=0}^n \gamma_k a_k b_k(x).$$

If the operator (1) is hyperbolicity preserving, then the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is called a *multiplier sequence for the basis B* , or a *B -multiplier sequence*.

The characterization of multiplier sequences for various orthogonal bases for $\mathbb{R}[x]$ and the research thereof has been of particular interest within the last many years. It appears this line of research was first developed by Turán (see [Turán 1952] and [Turán 1954]), who used this theory to investigate the famous Riemann hypothesis. For this reason, among others, the theory of multiplier sequences is being developed to this day. In this work, we focus our efforts on multiplier sequences for the Chebyshev polynomials of the first kind (we will simply refer to them as Chebyshev polynomials), $\{T_n\}_{n=0}^{\infty}$, which can be defined as the unique polynomials satisfying $T_n(\cos(\theta)) = \cos(n\theta)$ for $n \in \mathbb{N} \cup \{0\}$.

2. RELATED WORKS

This paper serves as a continuation of several works which analyze the multiplier sequences of Jacobi polynomials and/or special cases of these, such as the Legendre polynomials. [Bates 2014] contains a characterization of quadratic Jacobi (and, thus, Chebyshev) multiplier sequences. [Blakeman et al. 2012] contains an analysis of linear, quadratic, and geometric multiplier sequences for the Legendre polynomials. [Bunton et al. 2015] involves the study of complex zero decreasing operators and demonstrates the existence of a class of multiplier sequences for the Chebyshev basis. [Chasse et al. 2018] contains a proof that polynomial multiplier sequences for the Legendre basis must have the form $\{h(k^2 + k)\}_{k=0}^{\infty}$, where $h \in \mathbb{R}[x]$. [Forgács et al. 2014] contains a proof of the nonexistence of cubic multiplier sequences for the Legendre basis. [Forgács et al. 2012] looks more broadly at bases of simple sets and investigates when a set of multiplier sequences for one basis will be included in the

set of multiplier sequences for another basis. In [Yoshida 2013], it is shown that there are no linear multiplier sequences for the Jacobi basis. This collection of works provides a context for our research, as well as a standard of proofs and approaches to use, which we apply to the Chebyshev basis.

3. COMPUTATIONAL TOOLS

Mathematica, Sage, Wolfram Alpha, and the Python programming language were the central software elements used as investigative tools in this project. They were employed to discover and verify conjectures that led to our main results. All programming was written, compiled, and executed in a local development environment, but is public to view and to download on GitHub: https://github.com/joshshterenber/Chebyshev_MS.

4. THE SYMBOL OF A LINEAR OPERATOR

In this section, we develop a condition that must be met by a hyperbolicity preserving operator that will be useful to us later on. The argument is similar to that in section 3 (“Form and Order”) of [Chasse et al. 2018], which we outline here for the convenience of the reader.

For a general linear operator K , the symbol of K is given by

$$G_K(x, y) = \sum_{k=0}^{\infty} \frac{(-1)^k K[x^k]y^k}{k!} = K[e^{-xy}],$$

where the operator K acts on e^{-xy} as a function of x alone (see [Borcea and Brändén 2009] for a comprehensive treatment of the symbol of an operator as it relates to hyperbolicity preserving operators). Furthermore, any linear operator on $\mathbb{R}[x]$ can be represented as a differential operator. As in, for example, [Piotrowski 2007, Prop. 29], there is a unique set of real polynomials $\{Q_k\}$ such that

$$(2) \quad K = \sum_{k=0}^{\infty} Q_k(x)D^k \quad \left(D = \frac{d}{dx} \right).$$

It follows that the symbol G_K is given by

$$(3) \quad G_K(x, y) = \sum_{k=0}^{\infty} Q_k(x)D^k e^{-xy} = e^{-xy} \sum_{k=0}^{\infty} Q_k(x)(-1)^k y^k.$$

As noted in [Brändén and Ottergren 2014] and [Forgács et al. 2014], if the operator K is hyperbolicity preserving then we can act on the function G_K in equation (3) as a function of x alone by the multiplier sequence (for the standard basis) $\{1, 0, 0, 0, \dots\}$ and the resulting function,

$$(4) \quad G_K(0, y) = \sum_{k=0}^{\infty} Q_k(0)(-1)^k y^k,$$

must belong to the Laguerre-Pólya class, defined as follows: The *Laguerre-Pólya Class*, denoted $\mathcal{L} - \mathcal{P}$, is the set of entire functions φ which can be expressed in the form

$$\varphi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k} \right) e^{-x/x_k} \quad (0 \leq \omega \leq \infty),$$

where $b, c, x_k \in \mathbb{R}$, m is a non-negative integer, $a \geq 0$, $x_k \neq 0$, and $\sum_{k=1}^{\omega} x_k^{-2} < \infty$.

To obtain the values of $Q_k(0)$ in equation (4), we use the following lemma which is readily verified by calculating $K[x^n]$ and evaluating the resulting expression for $x = 0$.

Lemma 1. *For any differential operator*

$$K = \sum_{k=0}^{\infty} Q_k(x) D^k \quad \left(D = \frac{d}{dx} \right)$$

on $\mathbb{R}[x]$, the coefficient polynomials evaluated at 0 can be computed by

$$Q_k(0) = \frac{1}{k!} [K[x^k]]_{x=0} \quad (k = 0, 1, 2, \dots).$$

Thus, we arrive at the following proposition which will be central to our analysis of polynomial sequences of odd degree:

Proposition 2. *If K is a linear operator on $\mathbb{R}[x]$ and*

$$G_K(0, y) = \sum_{k=0}^{\infty} Q_k(0) (-1)^k y^k \quad \left(Q_k(0) = \frac{1}{k!} [K[x^k]]_{x=0} \right)$$

does not belong to the Laguerre-Pólya class, then K is not hyperbolicity preserving.

5. SPECIALIZING TO THE CHEBYSHEV POLYNOMIALS

We now specialize the results of the last section to investigate multiplier sequences for the Chebyshev polynomial basis.

Proposition 3. *Let $\{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers, and let K be the corresponding linear operator on $\mathbb{R}[x]$ defined by*

$$(5) \quad K \left[\sum_{k=0}^n a_k T_k(x) \right] = \sum_{k=0}^n \gamma_k a_k T_k(x),$$

where T_k denotes the k th Chebyshev polynomial. Let $Q_{2k}(0)$ be defined by

$$(6) \quad Q_{2k}(0) = \frac{2^{1-2k}}{(2k)!} \left[\frac{1}{2} \binom{2k}{k} \gamma_0 + \sum_{i=1}^k (-1)^i \binom{2k}{k-i} \gamma_{2i} \right].$$

If

$$G_K(0, y) = \sum_{k=0}^{\infty} Q_{2k}(0) y^{2k}$$

does not belong to the Laguerre-Pólya class, then K is not a hyperbolicity preserving operator. Thus, $\{\gamma_k\}_{k=0}^{\infty}$ is not a multiplier sequence for the Chebyshev basis.

Proof. Consider a sequence $\{\gamma_k\}_{k=0}^{\infty}$ with a corresponding operator K given by (5). Applying Lemma 1, we have

$$Q_n(0) = \frac{1}{n!} [K[x^n]]_{x=0}.$$

From the known expansion (see [Cody 1970, p. 412] or [Mason and Handscomb 2002, p. 22])

$$x^n = 2^{1-n} \left(\sum_{\substack{j=0 \\ j \equiv n \pmod{2}}}^n \binom{n-j}{2} T_j(x) \right),$$

where the prime on the sum means that the contribution of the $j = 0$ term (if any) is halved, we obtain

$$\begin{aligned} Q_n(0) &= \frac{1}{n!} \left[K \left[2^{1-n} \left(\sum_{\substack{j=0 \\ j \equiv n \pmod{2}}}^n \binom{n-j}{2} T_j(x) \right) \right] \right]_{x=0} \\ &= \frac{2^{1-n}}{n!} \left(\sum_{\substack{j=0 \\ j \equiv n \pmod{2}}}^n \binom{n-j}{2} \gamma_j T_j(0) \right). \end{aligned}$$

Using the fact (see, e.g., [Mason and Handscomb 2002, p. 24]) that

$$T_n(0) = \begin{cases} 0 & n \text{ odd,} \\ (-1)^{n/2} & n \text{ even,} \end{cases}$$

we see that $Q_n(0) = 0$ when n is odd. If n is even, say $n = 2k$, then

$$Q_{2k}(0) = \frac{2^{1-2k}}{(2k)!} \left[\frac{1}{2} \binom{2k}{k} \gamma_0 + \sum_{i=1}^k (-1)^i \binom{2k}{k-i} \gamma_{2i} \right].$$

The result now directly follows from Proposition 2. □

This leads to the following useful result.

Corollary 4. *With the notation of Proposition 3, if there exists $n \in \mathbb{N}$ such that $Q_{2n}(0)Q_{2n+2}(0) > 0$, then $\{\gamma_k\}_{k=0}^\infty$ is not a multiplier sequence for the Chebyshev basis.*

Proof. The function $G_K(0, y)$ in the previous lemma is an even function of y and an even function in the Laguerre-Pólya class must have coefficients that alternate in sign (see, for example, [Craven and Csordas 2004, Theorem 3.5]). □

5.1. Polynomially Interpolated Multiplier Sequences. We now focus on sequences of the form $\{p(k)\}_{k=0}^\infty$, where $p \in \mathbb{R}[x]$. This section contains three subsections regarding the quantities $Q_{2k}(0)$ in equation (6). The first deals with the contribution of any even powers. The second deals with the contribution of any odd powers. The third combines these for a general polynomial sequence and culminates in our main result.

5.1.1. Even powers. In this section, we appeal to the Chebyshev differential equation to get information on the contribution of even powers to the quantities $Q_{2k}(0)$.

Proposition 5. Let $n \in \mathbb{N} \cup \{0\}$ be an even integer and let K be the linear operator on $\mathbb{R}[x]$ defined by

$$K \left[\sum_{k=0}^n a_k T_k(x) \right] = \sum_{k=0}^n k^n a_k T_k(x).$$

Then the differential operator representation for K in equation (2) is of finite order. That is to say, $Q_k(x)$ is identically zero for all k sufficiently large.

Proof. Suppose $n = 2j$ for some $j \in \mathbb{N} \cup \{0\}$. Let

$$p(x) = \sum_{k=0}^n a_k T_k(x) \in \mathbb{R}[x].$$

Since the Chebyshev polynomials satisfy the differential equation (see, e.g., [Rainville 1960, p. 258 and p. 301] or [Mason and Handscomb 2002, p. 85])

$$k^2 T_k(x) = x T_k'(x) + (x^2 - 1) T_k''(x) \quad (k \in \mathbb{N} \cup \{0\}),$$

we have

$$K[p(x)] = \sum_{k=0}^n k^{2j} a_k T_k(x) = \sum_{k=0}^n (k^2)^j a_k T_k(x) = (xD + (x^2 - 1)D^2)^j p(x).$$

Thus, $K = (xD + (x^2 - 1)D^2)^j$, which is a finite order differential operator. \square

In general, any even power of k in a sequence of the form

$$\{\gamma_k\}_{k=0}^{\infty} = \{a_0 + a_1 k + a_2 k^2 + \cdots + a_n k^n\}_{k=0}^{\infty}$$

will only contribute a finite number of terms to the differential operator representation (2) for the corresponding operator defined by (5). Since our analysis will focus on the behavior of the sequence $Q_{2k}(0)$ for large indices, the even powered terms will not be relevant.

5.1.2. *Odd powers.* We begin by examining $Q_{2k}(0)$ for $\gamma_k = k^n$ where n a positive integer and we will eventually specialize to the case where n is odd.

Lemma 6. Let $n \in \mathbb{N}$ and let $\{\gamma_k\}_{k=0}^{\infty} = \{k^n\}_{k=0}^{\infty}$. Then the quantities $Q_{2k}(0)$ defined in equation (6) are given by

$$Q_{2k}(0) = \frac{2^{1-2k}}{(2k)!} A(n, k),$$

where

$$(7) \quad A(n, k) = \sum_{i=1}^k (-1)^i \binom{2k}{k-i} (2i)^n.$$

Proof. Note that, since $n \in \mathbb{N}$, we have that $\gamma_0 = 0$. The result now follows directly from equation (6) and our choice of γ_k . \square

Computer algebra systems give the following simplifications for these expressions:

$$A(1, k) = -\frac{k+1}{2k-1} \binom{2k}{k-1}$$

$$A(3, k) = \frac{4k(k+1)}{(2k-1)(2k-3)} \binom{2k}{k-1}$$

$$A(5, k) = \frac{-16k(k+1)(4k-1)}{(2k-1)(2k-3)(2k-5)} \binom{2k}{k-1}$$

Through a series of lemmas and an auxiliary function, we will show that, in general, $A(n, k)/\binom{2k}{k-1}$ is a rational function of k which is not identically zero whenever n is odd. We will make frequent use of the rising factorial, falling factorial, and hypergeometric function which are defined as follows:

The *rising and falling factorials* are defined, respectively, by

$$x^{(n)} = \prod_{k=0}^{n-1} (x+k) \quad \text{and} \quad (x)_n = \prod_{k=0}^{n-1} (x-k) \quad (n \in \mathbb{N}),$$

and $x^{(0)} = (x)_0 = 1$. The *hypergeometric function* is defined in terms of the rising factorial by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{a^{(n)} b^{(n)} z^n}{c^{(n)} n!}.$$

Lemma 7. *For any integer $k > 0$, the function*

$$f(x) = \sum_{i=1}^k \binom{2k}{k-i} x^i$$

can be represented using the hypergeometric function ${}_2F_1$ as

$$f(x) = x \cdot \binom{2k}{k-1} \cdot {}_2F_1(1, 1-k; 2+k; -x).$$

Proof. Using the definition of the hypergeometric function, re-indexing the sum, and using the fact that $(1-k)^{(i-1)} = 0$ for $i > k$, we get

$$\begin{aligned} x \cdot \binom{2k}{k-1} \cdot {}_2F_1(1, 1-k; 2+k; -x) &= \frac{x(2k)!}{(k-1)!(k+1)!} \sum_{i=0}^{\infty} \frac{1^{(i)}(1-k)^{(i)}(-x)^i}{(2+k)^{(i)} i!} \\ &= \frac{(2k)!}{(k-1)!(k+1)!} \sum_{i=1}^k \frac{(1-k)^{(i-1)}(-1)^{i-1}}{(2+k)^{(i-1)}} x^i. \end{aligned}$$

Simplifying the coefficient of x^i ,

$$\begin{aligned} \frac{(1-k)^{(i-1)}(-1)^{i-1}}{(2+k)^{(i-1)}} &= \frac{(1-k)(2-k)(3-k) \dots (i-1-k)(-1)^{i-1}}{(2+k)(3+k)(4+k) \dots (i+k)} \\ &= \frac{(k-1)(k-2)(k-3) \dots (k-i+1)}{(2+k)(3+k)(4+k) \dots (i+k)}. \end{aligned}$$

Dividing by $(k-1)!$ we have

$$\frac{1}{(k-1)!} \frac{(1-k)^{(i-1)}(-1)^{i-1}}{(2+k)^{(i-1)}} = \frac{1}{(2+k)(3+k)(4+k) \dots (i+k) \cdot (k-i)!} = \frac{(k+1)!}{(k+i)!(k-i)!}.$$

Dividing by $(k+1)!$ then gives

$$\frac{1}{(k-1)!(k+1)!} \frac{(1-k)^{(i-1)}(-1)^{i-1}}{(2+k)^{(i-1)}} = \frac{1}{(k+i)!(k-i)!}.$$

Therefore,

$$\begin{aligned} x \cdot \binom{2k}{k-1} \cdot {}_2F_1(1, 1-k; 2+k; -x) &= (2k)! \cdot \sum_{i=1}^k \frac{1}{(k-1)!(k+1)!} \frac{(1-k)^{(i-1)}(-1)^{i-1}}{(2+k)^{(i-1)}} x^i \\ &= \sum_{i=1}^k \frac{(2k)!}{(k+i)!(k-i)!} x^i = \sum_{i=1}^k \binom{2k}{k-i} x^i. \end{aligned}$$

□

The function in the previous lemma is related to $A(n, k)$ through differentiation. We record this in the following lemma.

Lemma 8. *Let $k \in \mathbb{N}$ and let*

$$f(x) = \sum_{i=1}^k \binom{2k}{k-i} x^i.$$

The quantity $A(n, k)$ from equation (7) is given by

$$A(n, k) = 2^n [\theta^n f(x)]_{x=-1} \quad \left(\theta = xD = x \frac{d}{dx} \right).$$

Proof. From the relation $\theta^n x^i = i^n x^i$, and the linearity of the operator θ , we obtain

$$\begin{aligned} 2^n [\theta^n f(x)]_{x=-1} &= 2^n \left[\theta^n \sum_{i=1}^k \binom{2k}{k-i} x^i \right]_{x=-1} = 2^n \left[\sum_{i=1}^k \binom{2k}{k-i} i^n x^i \right]_{x=-1} \\ &= 2^n \sum_{i=1}^k \binom{2k}{k-i} i^n (-1)^i = A(n, k). \end{aligned}$$

□

Combining the previous two lemmas, we obtain:

Corollary 9. *For positive integers n and k ,*

$$A(n, k) = 2^n \binom{2k}{k-1} [\theta^n (x \cdot {}_2F_1(1, 1-k; 2+k; -x))]_{x=-1} \quad \left(\theta = x \frac{d}{dx} \right).$$

Now, in order to simplify the derivatives in the previous corollary, we define a new function of the variable x with parameters n and k by

$$(8) \quad g(n, k; x) = x^{n+1} \cdot {}_2F_1(1+n, 1+n-k; 2+n+k; -x)$$

and note that the conclusion of Corollary 9 can be stated as

$$(9) \quad A(n, k) = 2^n \binom{2k}{k-1} [\theta^n g(0, k; x)]_{x=-1}.$$

When calculating a related expression for the first few values of n , we obtained:

$$\theta g(0, k; x) = g(0, k; x) + \frac{k-1}{k+2}g(1, k; x)$$

$$\theta g(1, k; x) = 2g(1, k; x) + 2\frac{k-2}{k+3}g(2, k; x)$$

$$\theta g(2, k; x) = 3g(2, k; x) + 3\frac{k-3}{k+4}g(3, k; x)$$

and this pattern generalizes as shown in the following lemma. As the arguments of k and x do not change, we will write $g(n) = g(n, k; x)$ for clarity.

Lemma 10. *Let*

$$g(n) = x^{n+1} \cdot {}_2F_1(1+n, 1+n-k; 2+n+k; -x).$$

Then

$$\theta g(n) = (n+1) \left(g(n) + \frac{k-n-1}{k+n+2}g(n+1) \right) \quad \left(\theta = x \frac{d}{dx} \right).$$

Proof. We calculate

$$\begin{aligned} \theta g(n) &= x \frac{d}{dx} [x^{n+1} \cdot {}_2F_1(1+n, 1+n-k; 2+n+k; -x)] \\ &= (n+1)x^{n+1} {}_2F_1(1+n, 1+n-k; 2+n+k; -x) \\ &\quad + x^{n+2} \frac{d}{dx} ({}_2F_1(1+n, 1+n-k; 2+n+k; -x)) \\ &= (n+1)g(n) - x^{n+2} \frac{(1+n)(1+n-k)}{2+n+k} {}_2F_1(2+n, 2+n-k; 3+n+k; -x) \\ &= (n+1) \left(g(n) + \frac{k-n-1}{2+n+k}g(n+1) \right), \end{aligned}$$

where we have used the identity (see, e.g., [Rainville 1960, p. 69])

$$\frac{d}{dx} F(a, b; c; x) = \frac{ab}{c} F(a+1, b+1; c+1; x).$$

□

Using Lemma 10 to recursively calculate $\theta^n g(n, k, 1)$, we obtained constants for each term which appear in the Online Encyclopedia of Integer Sequences [OEIS, A028246] as Worpitzky's triangular array involving alternating binomial power sums. From this we obtained the following lemma. Again, we will suppress the k and x from the function g for clarity.

Lemma 11. *For positive integers n and k , we have*

$$\theta^n g(0) = \sum_{i=0}^n \frac{(k-1)_i}{(2+k)^{(i)}} g(i) \cdot C_{i,n} \quad \left(\theta = x \frac{d}{dx} \right),$$

where $g(n) = g(n, k; x)$ is defined in equation (8) and $C_{i,n}$ is Worpitzky's triangular array defined by

$$(10) \quad C_{i,n} = \frac{1}{i+1} \sum_{j=0}^{i+1} (-1)^{i-j+1} \binom{i+1}{j} j^{n+1}.$$

Proof. (By induction.) The result is readily verified for $n = 0$. Furthermore, the case $n = 1$ can be obtained directly from Lemma 10. Now suppose the lemma holds for some integer $n \geq 0$. Then

$$\begin{aligned} \theta^{n+1}g(0) &= \theta \left(\sum_{i=0}^n \frac{(k-1)_i}{(2+k)^{(i)}} g(i) \cdot C_{i,n} \right) \\ &= \left(\sum_{i=0}^n \frac{(k-1)_i}{(2+k)^{(i)}} \theta g(i) \cdot C_{i,n} \right) \\ &= \sum_{i=0}^n \frac{(k-1)_i}{(2+k)^{(i)}} (i+1)g(i) \cdot C_{i,n} + \sum_{i=0}^n \frac{(k-1)_i}{(2+k)^{(i)}} (i+1) \frac{k-i-1}{k+i+2} g(i+1) \cdot C_{i,n} \\ &= \sum_{i=0}^n \frac{(k-1)_i}{(2+k)^{(i)}} (i+1)g(i) \cdot C_{i,n} + \sum_{i=1}^{n+1} \frac{(k-1)_i}{(2+k)^{(i)}} g(i) \cdot iC_{i-1,n} \\ &= \sum_{i=0}^{n+1} \frac{(k-1)_i}{(2+k)^{(i)}} g(i) \cdot C_{i,n+1} \end{aligned}$$

where, for $0 \leq i < n+1$ we have used the identities (see, e.g., [Worpitzky 1883, p. 210])

$$C_{i,n+1} = (i+1)C_{i,n} + iC_{i-1,n}$$

and

$$C_{n+1,n+1} = (n+1)C_{n,n}.$$

□

We will need to evaluate these expressions at $x = -1$. The next lemma aids us in this endeavour.

Lemma 12. *Let $i, k \in \mathbb{N} \cup \{0\}$. If $k > i/2$, then*

$$g(i, k; -1) = (-1)^{i+1} \frac{(k+1)^{(i+1)}}{(2k)_{i+1}}.$$

Proof. By definition (see equation (8))

$$g(i, k; -1) = (-1)^{i+1} {}_2F_1(1+i, 1+i-k; 2+i+k; 1).$$

Using the formula ([Rainville 1960, p. 49, l. 7])

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\operatorname{Re}(c-a-b) > 0)$$

we obtain for $k > i/2$

$$g(i, k; -1) = (-1)^{i+1} \frac{\Gamma(2+i+k)\Gamma(2k-i)}{\Gamma(1+k)\Gamma(2k+1)}$$

$$\begin{aligned}
&= (-1)^{i+1} \frac{(1+i+k)!(2k-i-1)!}{k!(2k)!} \\
&= (-1)^{i+1} \frac{(k+1)^{(i+1)}}{(2k)_{i+1}}.
\end{aligned}$$

□

We may now combine the preceding results to obtain the following proposition.

Proposition 13. *For positive integers k and n satisfying $k > n/2$, the expressions $A(n, k)$ defined in equation (7) are given by*

$$A(n, k) = 2^n \binom{2k}{k-1} (k+1) \frac{N(n, k)}{(2k)_{n+1}},$$

where

$$N(n, k) = \sum_{i=0}^n (-1)^{i+1} C_{i,n} (k-1)_i (2k-i-1)_{n-i},$$

and $C_{i,n}$ is defined as in equation (10). In particular, we note that $A(n, k) / \binom{2k}{k-1}$ is a rational function of the variable k .

Proof. Combining equation (9), Lemma 11, and Lemma 12, we obtain

$$\begin{aligned}
A(n, k) &= 2^n \binom{2k}{k-1} [\theta^n g(0, k; x)]_{x=-1} \\
&= 2^n \binom{2k}{k-1} \sum_{i=0}^n \frac{(k-1)_i}{(2+k)^{(i)}} g(i, k; -1) \cdot C_{i,n} \\
&= 2^n \binom{2k}{k-1} \sum_{i=0}^n \frac{(k-1)_i}{(2+k)^{(i)}} (-1)^{i+1} \frac{(k+1)^{(i+1)}}{(2k)_{i+1}} \cdot C_{i,n} \\
&= 2^n \binom{2k}{k-1} \sum_{i=0}^n (-1)^{i+1} \frac{(k-1)_i (k+1)}{(2k)_{i+1}} \cdot C_{i,n}.
\end{aligned}$$

Finally, the result follows from rewriting the sum as a single fraction using the common denominator $(2k)_{n+1}$. □

We summarize this section in the following corollary.

Corollary 14. *Let n be a positive integer and, for $k \in \mathbb{N} \cup \{0\}$, define $\gamma_k = k^n$. Then, for $k > n/2$, the quantities $Q_{2k}(0)$ defined in equation (6) are given by*

$$Q_{2k}(0) = \frac{2^{1-2k+n}}{(2k)!} \binom{2k}{k-1} (k+1) \frac{N(n, k)}{(2k)_{n+1}},$$

where

$$N(n, k) = \sum_{i=0}^n (-1)^{i+1} C_{i,n} (k-1)_i (2k-i-1)_{n-i},$$

and $C_{i,n}$ is defined as in equation (10).

Proof. Combine Lemma 6 with Proposition 13. □

We note that, in light of section 5.1.1, when n is even, the quantities $Q_{2k}(0)$ must be zero for all sufficiently large k . Thus, the rational function that appears must reduce to the identically zero function, i.e., $N(n, k)$ must be identically zero. When n is odd, this is not the case. We record this in the following lemma that will be used in the main result of the next section.

Lemma 15. *If n is odd, then $N(n, n/2) \neq 0$, where N is defined in Corollary 14.*

Proof. We have

$$N(n, n/2) = (-1)^{n+1} C_{n,n} \left(\frac{n}{2} - 1\right) \left(\frac{n}{2} - 2\right) \left(\frac{n}{2} - 3\right) \cdots \left(\frac{n}{2} - n\right).$$

Since (see, e.g., [Worpitzky 1883, p. 225]) $C_{n,n} = n! \neq 0$ and n is odd, $N(n, n/2) \neq 0$ (Note, if n is even, then the quantity is zero, as expected). \square

5.1.3. *Main Results.* We are now in a position to state and prove our main result.

Theorem 16. *Suppose*

$$\{\gamma_k\}_{k=0}^{\infty} = \{k^n + b_1 k^{n-1} + b_2 k^{n-2} + \cdots + b_n\}_{k=0}^{\infty} \quad (n \in \mathbb{N} \cup \{0\}, b_i \in \mathbb{R})$$

is a multiplier sequence for the Chebyshev basis. Then n is even and $a_i = 0$ whenever i is odd. I.e., the polynomial interpolating the sequence γ_k must have no odd powers. It must be an even function.

Proof. We focus our attention on the sequence $Q_{2k}(0)$ for large values of k . As seen in section 5.1.1, we may omit any even powers in the sequence $\{\gamma_k\}_{k=0}^{\infty}$. What remains is a sequence of the form

$$\gamma_k = k^n + b_2 k^{n-2} + b_4 k^{n-4} + \cdots + b_{n-1} k,$$

where we are assuming that n is odd. For this sequence, equation (6) becomes

$$Q_{2k}(0) = \frac{2^{1-2k}}{(2k)!} \left[\sum_{i=1}^k (-1)^i \binom{2k}{k-i} \gamma_{2i} \right]$$

or, in light of Lemma 6,

$$Q_{2k}(0) = \frac{2^{1-2k}}{(2k)!} [A(n, k) + b_2 A(n-2, k) + b_4 A(n-4, k) + \cdots + b_{n-1} A(1, k)].$$

Applying Proposition 13, we obtain

$$Q_{2k}(0) = \frac{2^{1-2k}}{(2k)!} \binom{2k}{k-1} (k+1) \left[\frac{2^n}{(2k)_{n+1}} N(n, k) + \frac{2^{n-2} b_2}{(2k)_{n-1}} N(n-2, k) + \cdots + \frac{2^1 b_{n-1}}{(2k)_1} N(1, k) \right]$$

Factoring out $(2k)_{n+1}$, we see that the sign of $Q_{2k}(0)$ will be completely determined by the expression

$$2^n N(n, k) + 2^{n-2} (2k-n)(2k-n+1) b_2 N(n-2, k) + \cdots + 2^1 (2k-n)^{(n-1)} b_{n-1} N(1, k),$$

which is a polynomial in the variable k . We claim that this polynomial is not identically zero. Indeed, by setting $k = n/2$ the expression reduces to $2^n N(n, n/2)$ which is non-zero by Lemma 15. Therefore, for all sufficiently large values of k , this polynomial in k must be strictly positive or strictly negative. It follows that the same is true of the sequence $Q_{2k}(0)$. Therefore, the result follows from Corollary 4. \square

We can now draw some conclusions about geometric sequences as well. First, we need two known results.

Theorem 17. [Forgács et al. 2012, Theorem 2] *Let $B = \{b_k(x)\}_{k=0}^{\infty}$ and $Q = \{q_k(x)\}_{k=0}^{\infty}$ be simple sets of polynomials. If there exists an $\alpha > 1$ such that both $\{\alpha^k\}_{k=0}^{\infty}$ and $\{\alpha^{-k}\}_{k=0}^{\infty}$ are B -multiplier sequences, then every Q -multiplier sequence is also a B -multiplier sequence.*

Theorem 18. [Chasse et al. 2018, Theorem 2] *If a multiplier sequence for the Legendre basis can be interpolated by a polynomial p , then $p(x) = h(x^2 + x)$ for some polynomial $h \in \mathbb{R}[x]$.*

Since there are clearly multiplier sequences for the Legendre basis which are not multiplier sequences for the Chebyshev basis (and vice versa) we can conclude that there does not exist an $\alpha > 1$ such that both $\{\alpha^k\}_{k=0}^{\infty}$ and $\{\alpha^{-k}\}_{k=0}^{\infty}$ are multiplier sequences for the Chebyshev basis. In fact, much more is true. We detail this in our next theorem which characterizes geometric multiplier sequences for the Chebyshev basis.

Theorem 19. *Let $r \in \mathbb{R}$. Then $\{r^k\}_{k=0}^{\infty}$ is a multiplier sequence for the Chebyshev basis if and only if $r \in \{-1, 0, 1\}$.*

Proof. Let K be the operator corresponding to the sequence $\{r^k\}_{k=0}^{\infty}$ as defined in (5). A calculation shows that

$$\begin{aligned} K[(x + r/3)^3] &= K \left[\frac{1}{4}T_3(x) + \frac{r}{2}T_2(x) + \left(\frac{3}{4} + \frac{r^2}{3}\right)T_1(x) + \left(\frac{r}{2} + \frac{r^3}{27}\right)T_0(x) \right] \\ &= \frac{r^3}{4}T_3(x) + \frac{r^3}{2}T_2(x) + \left(\frac{3r}{4} + \frac{r^3}{3}\right)T_1(x) + \left(\frac{r}{2} + \frac{r^3}{27}\right)T_0(x) \\ &= r^3x^3 + r^3x^2 + \left(\frac{3}{4}r - \frac{5}{12}r^3\right)x + \frac{1}{2}r - \frac{25}{54}r^3. \end{aligned}$$

It is known (see, e.g., [Cajori 1904, p. 36-40]) that a cubic function $ax^3 + bx^2 + cx + d$ has non-real zeros if and only if the quantity

$$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$

is negative. For $K[(x + r/3)^3]$, we obtain $\Delta = -27r^6(r^2 - 1)^2/16$. It follows that if $\{r^k\}_{k=0}^{\infty}$ is a multiplier sequence for the Chebyshev basis, then $r \in \{-1, 0, 1\}$.

Conversely, the sequences $\{0^k\}_{k=0}^{\infty}$ and $\{1^k\}_{k=0}^{\infty}$ are clearly multiplier sequences for any basis. The sequence $\{(-1)^k\}_{k=0}^{\infty}$ is also a multiplier sequence for the Chebyshev basis as a consequence of the symmetry relation $T_n(-x) = (-1)^nT_n(x)$ (see, e.g., [Mason and Handscomb 2002, p. 2-3]. Indeed, the corresponding operator K in this case satisfies $T[p(x)] = p(-x)$ and so $T[p]$ has the same number of real zeros as p . \square

6. CONCLUSIONS AND FURTHER WORK

We have demonstrated that if a multiplier sequence for the Chebyshev basis can be interpolated by a polynomial, then the polynomial must be an even function. This result closely mirrors one that was obtained for the Legendre polynomials in [Chasse et al. 2018]. The next point of research beyond this could be to analyze the more general Gegenbauer and Jacobi polynomials (see, for example, [Rainville 1960] for the definition of these polynomials) in the same fashion. In particular, based on the results for the Legendre and Chebyshev bases, we pose the following problem.

Problem 20. *Is it true that if a multiplier sequence for the Jacobi polynomial basis $\{P_k^{(\alpha,\beta)}\}_{k=0}^\infty$ can be interpolated by a polynomial p , then there must exist $h \in \mathbb{R}[x]$ such that $p(x) = h(x(x + \alpha + \beta + 1))$?*

One could also attempt to determine which even polynomials p generate a multiplier sequence for the Chebyshev basis. To date, and to the best of our knowledge, only the quadratic multiplier sequences have been characterized for the Chebyshev basis. Finally, one could attempt to find a complete characterization of multiplier sequences for the Jacobi polynomial basis.

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REFERENCES

- [Bates 2014] R. Bates, *Hyperbolicity Preserving Differential Operators and Classifications of Orthogonal Multiplier Sequences*, Ph.D. Dissertation, University of Hawai'i at Manoa, 2014. <http://hdl.handle.net/10125/101054>
- [Blakeman et al. 2012] K. Blakeman, E. Davis, T. Forgács, and K. Urabe, “On Legendre Multiplier Sequences”, *Missouri J. Math. Sci.* **24**:1 (2012), 7-23. <https://doi.org/10.35834/MJMS%2F1337950496>
- [Borcea and Brändén 2009] J. Borcea and P. Brändén, “Pólya-Schur master theorems for circular domains and their boundaries”, *Ann. of Math. (2)* **170**:1 (2009), 465-492. <https://doi.org/10.4007/annals.2009.170.465>
- [Brändén and Ottergren 2014] P. Brändén and E. Ottergren, “A Characterization of Multiplier Sequences for Generalized Laguerre Bases”, *Constr. Approx.* **39**:3 (2014), 585-596. <https://doi.org/10.1007/s00365-013-9204-4>
- [Bunton et al. 2015] A. Bunton, N. Jacobs, S. Jenkins, C. McKenry, A. Piotrowski and L. Scott, “Non-real zero decreasing operators related to orthogonal polynomials”, *Involve* **8**:1 (2015), pp. 129-146. <https://doi.org/10.2140/involve.2015.8.129>
- [Cajori 1904] F. Cajori, *An introduction to the modern theory of equations*, Macmillan, London, 1904.
- [Chasse et al. 2018] M. Chasse, T. Forgács, and A. Piotrowski, “Polynomially interpolated Legendre multiplier sequences”, *Comput. Methods and Funct. Theory* **18**:2 (2018), pp. 315-333. <https://doi.org/10.1007/s40315-017-0221-3>
- [Cody 1970] W. L. Cody, “A Survey of Practical Rational and Polynomial Approximation of Functions”, *SIAM Review*, **12**:3 (Jul., 1970), pp. 400-423. <https://doi.org/10.1137/1012082>
- [Craven and Csordas 2004] T. Craven and G. Csordas, “Composition theorems, multiplier sequences and complex zero decreasing sequences”, in *Value distribution theory and related topics*, Adv. Complex Anal. Appl. **3** 131-166, Kluwer Acad. Publ., Boston, MA, 2004. https://doi.org/10.1007/1-4020-7951-6_6
- [Forgács et al. 2014] T. Forgács and J. Haley and R. Menke and C. Simon, “The non-existence of cubic Legendre multiplier sequences”, *Involve* **7**:6 (2014), 773-786. <https://doi.org/10.2140/involve.2014.7.773>
- [Forgács et al. 2012] T. Forgács, J. Tipton, and B. Wright, “Multiplier sequences for simple sets of polynomials”, *Acta Math. Hungar.* **137**:4 (2012), 282-295 <https://doi.org/10.1007/s10474-012-0222-7>
- [Mason and Handscomb 2002] J.C. Mason and D.C. Handscomb, *Chebyshev Polynomials*, CRC Press, Boca Raton, Florida, 2002.
- [OEIS] OEIS Foundation Inc. (2019), The On-Line Encyclopedia of Integer Sequences, <http://oeis.org/A028246>.
- [Piotrowski 2007] A. Piotrowski, *Linear Operators and the Distribution of Zeros of Entire Functions*, Ph.D. Dissertation, University of Hawai'i at Manoa, 2007. <http://hdl.handle.net/10125/25932>
- [Rainville 1960] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960.

- [Turán 1952] P. Turán, “Sur l’algèbre fonctionnelle”, *Compt. Rend. du prem. (Août-2 Septembre 1950) Congr. des Math. Hongr.* **27**, Akadémiai Kiadó, Budapest (1952), 279-290.
- [Turán 1954] P. Turán, “Hermite-expansion and strips for zeros of polynomials”, *Arch. Math.* **5** (1954), 148-152. <https://doi.org/10.1007/BF01899331>
- [Worpitzky 1883] J. Worpitzky, “Studien über die Bernoullischen und Eulerschen Zahlen”, *J. Reine Angew. Math.* **94** (1883), 203-232. http://resolver.sub.uni-goettingen.de/purl?PPN243919689_0094
- [Yoshida 2013] R. Yoshida, *Linear and Non-Linear Operators, and the Distribution of Zeros of Entire Functions*, Ph.D. Dissertation, University of Hawai’i at Manoa, 2013. <http://hdl.handle.net/10125/29455>