Developing Entanglement Verification for Large Bipartite Systems

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ABSTRACT

It is a major problem within the field of Quantum Information whether the state of a system is entangled or separable. Exact methods have been developed for two qubit systems, but for larger systems determining the boundary between separable and entangled states has been shown to be NP-hard. An iterative method has been proposed that creates progressively tighter bounds from each direction, but this method is resource limited, and determining the boundary precisely requires an infinite number of iterations. Further, it is also believed that successive iterations scale badly. Instead, a weighted measure is developed over the delineated subspaces, producing a likelihood function by convex optimization. This presentation describes progress towards understanding and deploying this method in support of experimental work involving entangled states.

INTRODUCTION

The field of Quantum Information has developed many promising applications for the use of entangled states such as classic cryptography, superdense communication, and teleportation. Entanglement has been referred to as "a resource as real as energy [1]," and the use of this resource is dependent on our ability to characterize, control, and quantify it [1]. One of the major roadblocks to implementing these theoretical methods as realizable devices in the lab is the determination of separability in the state of an experimental system. Without confirmation that a system is entangled, any subsequent experimental results must be called into question if the interpretation of the results proceeds under the assumption of the presence of entanglement.

Entanglement is a quantum mechanical property where physically separated objects share a single quantum state. An interaction with one of these objects will affect the entire quantum state, including other entangled objects. Systems that are not entangled are known as separable, and do not have any of the special characteristics of entanglement. The determination of separability and entanglement is mutually exclusive, either a system is entangled or it is separable, not both or neither. Separable states may be represented as a mixture of product states, whereas entangled states cannot be. Entanglement need not be complete; systems may be only partially entangled and still not be separable. Entanglement is the determining factor; an admixture of a separable state and an entangled state – known as partially entangled – is still entangled, because it cannot be separated.

It has been shown that directly determining if the state of all but the simplest systems is separable is an NP-hard problem, and is thus not amenable to classical methods. Getting an exact solution to this problem is very difficult, and the development of even approximate solutions is both delicate and imprecise.

First, the set of all states within a system is convex. Furthermore, the subset of all separable states within a system is also convex. This allows us to describe the problem within the language of convex optimization. The mathematics involved in this area has been explored thoroughly.

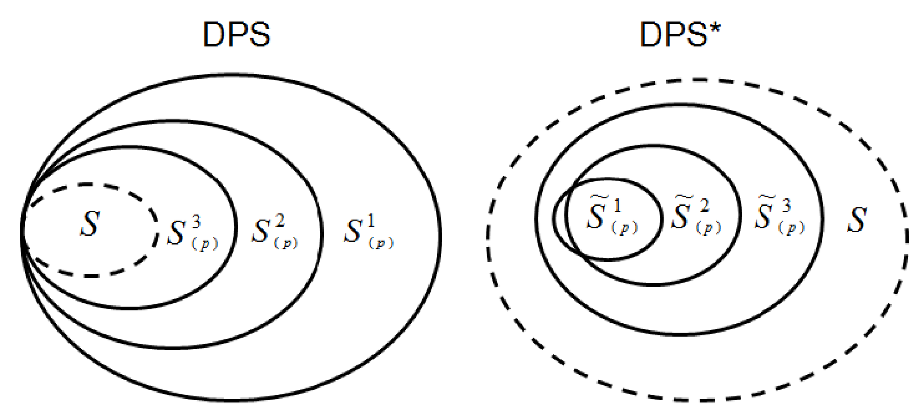
Another key aspect in finding a potential method of solution to this problem was proposed by Peres. The density matrix for any physical state must be positive. For separable states, the partial transpose of the density matrix is also always positive. The converse, for entangled states, does not always obtain. This result forms the basis for the positive partial transpose (PPT) test [2].

PPT is exact in the case of two qubits, but for large systems (such as two qutrits) this test is no longer exact. For any system larger than two qubits, a negative PPT will still imply an entangled state, but a positive PPT will no longer imply a separable state.

METHODS

There is a well defined boundary between the set of separable and the set of entangled states in a system, but it is impossible to directly determine this boundary for systems larger than two qubits. The direct method, known as the positive partial transpose (PPT), is only exact in the two smallest systems: two qubit systems, and a system composed of one qubit and one qutrit [1].

Doherty, Parrilo, and Spedalieri [3] have observed that, given a separable bipartite density matrix, one may construct a new n-partite state consisting of the original bipartite state and a number of "symmetric extensions." This idea starts with a single, two part state with parts A and B. Extensions of the A part can be found (referred to as A') such that when a partial trace is done over the new three part system that traces out A, the resulting bipartite density matrix is identical to the original. This can continue, and an infinite number of extensions can be found of A; any number of the extensions of A and the original A part itself can be traced out and the result will be the original bipartite density matrix. This holds true for separable states, but there comes a point with an entangled state where tracing out additional extensions leads to a nonphysical density matrix.



$$S^1_{(p)} \supseteq S^2_{(p)} \supseteq \ldots \supseteq S \qquad \tilde{S}^N, \tilde{S}^N_p \subseteq S, \forall N$$

$$\lim_{N\to\infty} S^N, S^N_p = S \qquad (\lim_{N\to\infty} \tilde{S}^N, \tilde{S}^N_p) = S$$

The bound between the set of separable states and the set of entangled states is approached both from in the outer and inner bound, from the DPS and the DPS* protocols, respectively. [4]

$$\rho_{A,A',A'',A''',\ldots B}$$

A density matrix with an infinite number of symmetric extensions

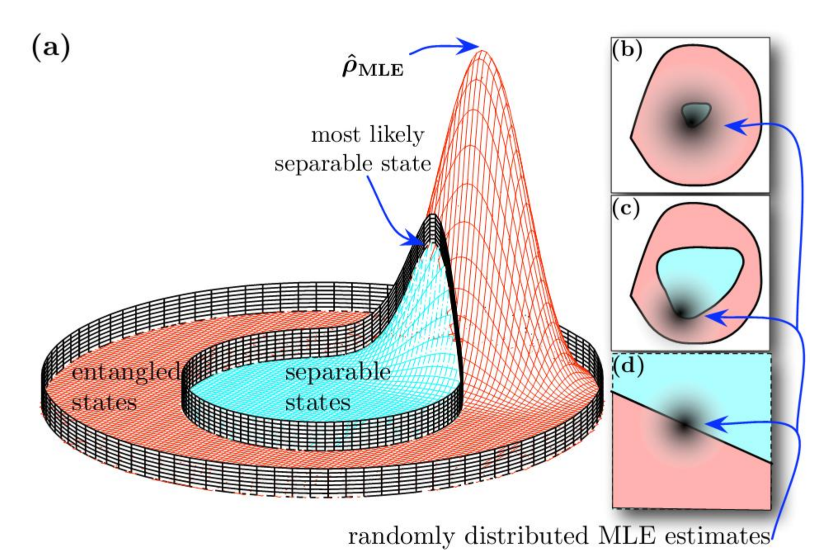
$$Tr_{A,A',A'',\ldots A^{n-1}}[\rho_{A,A',A'',A''',\ldots A^n,B}] = \rho_{A^n,B} = \rho_{A,B}$$

Tracing out all but one extension and one of the original parties wields the original state if the state was separable. This equation does not hold if the state was entangled.

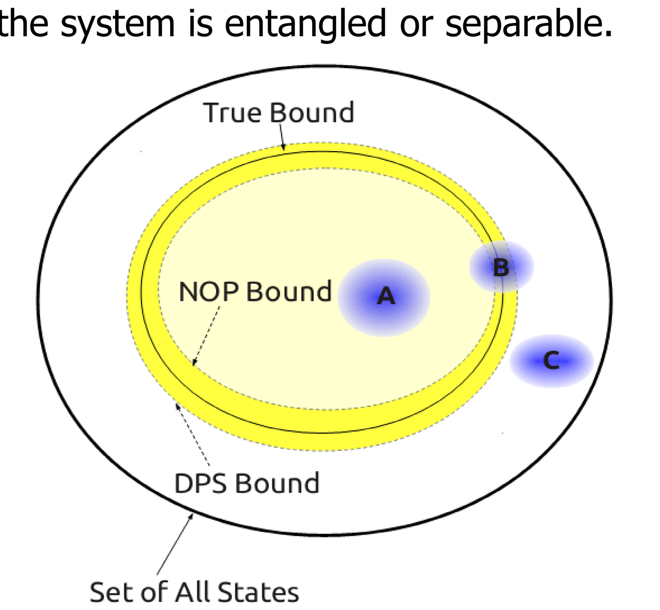
This method has the benefit of being iterative, as new extensions can be found one after another until either an extension is not found that returns a nonphysical density matrix or the method runs out of resources. With each iteration a boundary is put on the set of separable states that provides a progressively tighter outer bound on the true boundary between the set separable and the set entangled states.

An inner boundary can be found on the set of entangled states by a similar method developed by Navascues, Owari, and Plenio [5]. This method starts with the identity matrix and adds on a small amount of the target density matrix. This characterizes the distance from the density matrix to the set entangled states. Since the identity matrix is in itself separable, by slowly adding more and more of a potentially entangled state, we can evaluate the distance from the state to the entangled-separable boundary from within the set of separable states, instead of from outside. This, like the former method, is iterative, and provides a resource limited inner bound to the boundary between the set of separable and set of entangled states. The combination of these two methods provides both an inner and outer bound to that boundary.

Assuming that the boundary between the set of separable states and the set of entangled states is known, we could determine both the maximum likelihood over the set of all states and the maximum likelihood over the set of all separable states. Comparing these two maximum likelihoods creates a test of likely entanglement, originally proposed by Bloom-Kohout, Yin, and Enk [6]. The near neighborhood of the maximum likelihood can lie entirely within the set of entangled states, entirely within the set of separable states, or it can straddle the boundary between those two sets; see the second figure at left. Corresponding to the first case, the maximum likelihood over the set of separable states is significantly less than the maximum likelihood over the set of all states, and so we can (to some degree of confidence) conclude that the state in question was entangled. Conversely, in both the second and third cases we will find that the maximum likelihood over the set of all states is similar to the maximum likelihood over the set of separable states, we cannot (again, to some degree of confidence) conclude that the state in question is entangled. Note that this test is emphatically statistical; we can never clearly identify a single state as corresponding to a set of measurement outcomes, and thus we can never conclude definitively that the state of the system is entangled or separable.



A likelihood ratio over the set of separable states and all states of a given system. The maximally likely separable state can be compared to an experimentally determined state. [6]



Blue likelihood functions that fall into the yellow boundary area are unable to grant us clear evaluations on the separability of a given state

However, in this case, the DPS and NOP methods give us inner and outer bounds to the boundary between the set of separable states and the set of entangled states. Now we will have three maximum likelihoods. The set of states contained within the NOP (inner) boundary contains no entangled states, but may not contain all separable states. The set of states contained within the DPS (outer) boundary contains all separable states, but may also contain some entangled states. Thus there is an additional hierarchy of likelihoods, which follows the same pattern as above, and leads to the same set of conclusions, with the complication that the inner boundary likelihood may result in false positives (i.e. erroneous indication of entanglement), and the outer boundary likelihood may result in false negatives (i.e. erroneous rejection of the hypothesis of entanglement.) Since finding the true boundary between the set of entangled and the set of separable states is resource limited, our confidence in determining the entanglement of a given density matrix is dependent on how well we can define the overlap area of both the DPS and NOP methods. This allows us to make three possible evaluations: 1) It is likely that the state in question was separable, 2) It is likely that the state in question was entangled, or 3) The data given is consistent with both separable and entangled states, and a confident evaluation cannot be made.

Before a likelihood function can be maximized, we must first be able to find the symmetric extensions to develop the original bound proposed by Doherty, Parrilo, and Spedalieri [3]. To achieve this, we propose the use of semidefinite programming to search for these extensions.

RESULTS

Many tools exist for semidefinite programming, but each has specific drawbacks that may constrain finding a solution to this problem. Since in order to determine if a given state is entangled by the DPS test a potentially infinite number of extensions needed is needed, and since finding new extensions appears to scale badly, efficiency will be a major concern. Three possible routes have been observed.

MatLab
- Benefits: Scripted language; a wide variety of convex optimization solvers.
- Drawbacks: Cost; third party packages; third party solvers are no longer maintained and do not necessarily work well with current versions of MatLab; third party packages are not supported and documentation is restricted to a user's guide and some tests and benchmarks.

Python
- Benefits: Cost (free), able to use as a scripted language.
- Drawbacks: One solver; solver is not documented or supported; significant learning curve.

C/C++
- Benefits: Cost (generally free); lots of resources and documentation for low level functionality, prior art is known to exist but has not been publicly shared; a small number of solvers to choose from.
- Drawbacks: Not as robust compared to alternatives.

CONCLUSIONS

Semidefinite programming is known to be an effective tool in finding symmetric extensions, and theoretically there are many ways to develop solutions for the larger problem.

While the use of semidefinite programming will be important in the future, it is not going to move us toward a solution on its own. The use and implementation of a likelihood function over the separable states is required to find an answer that can be tested, and the development of this likelihood function may ultimately decide what tool is used for finding symmetric extensions.

Developing working code for a likelihood function can be done with any of the three tools that have been explored. Once a routine has been created, optimization can be done to make the solution more efficient.

REFERENCES

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. Reviews of Modern Physics, 81(2), 865 (2009).
[2] A. Peres. Separability Criterion for Density Matrices. Phys. Rev. Lett. 77, 1413 (1996).
[3] A.C. Doherty, P.A. Parrilo, and F.M. Spedalieri. Complete family of separability criteria. Phys. Rev. A, 69(2), 022308 (2004).
[4] M. Navascues, M. Owari, M. B. Plenio. A complete criterion for separability detection. Phys. Rev. Lett. 103, 160404 (2009).
[5] M. Navascues, M. Owari, and M.B. Plenio. The power of symmetric extensions for entanglement detection. Phys. Rev. A 80, 052306 (2009).
[6] R. Bloom-Kohout, J. Yin, S. Enk. Entanglement verification with finite data. Phys. Rev. Lett. 105, 170501 (2010).

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