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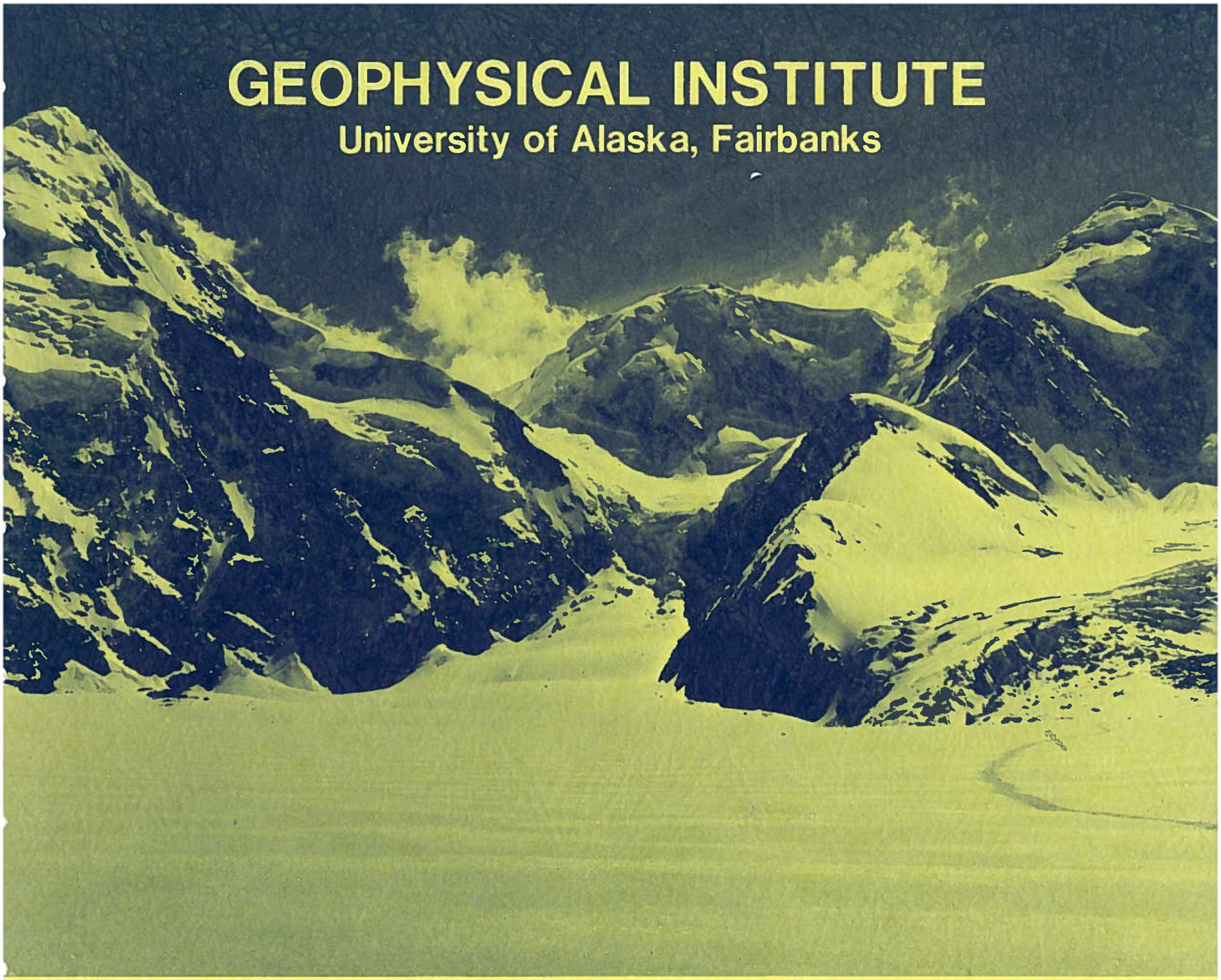

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ON THE DISCRETE ORDINATE METHOD FOR RADIATIVE TRANSFER CALCULATIONS IN ANISOTROPICALLY SCATTERING ATMOSPHERES

by

KNUT STAMNES

SCIENTIFIC REPORT
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ABSTRACT

The difficulties inherent in the conventional numerical implementation of the discrete ordinate method (following Chandrasekhar's prescription) for solving the radiative transfer equation are discussed. A matrix formulation is developed to overcome these difficulties, and it is specifically shown that the order of the algebraic eigenvalue problem can be reduced by a factor of 2. This results in considerable reduction of computing time, especially if high-order discrete ordinate solutions are desired. A new expression for the source function is derived and used to obtain angular distributions. By appealing to the reciprocity principle it is shown that substantial computational shortcuts are possible if only integrated quantities such as albedo and transmissivity are required. Comparison of fluxes calculated by the present approach with those obtained by other methods shows that low-order discrete ordinate approximations yield very accurate results. Thus, the present approach offers an efficient and reliable computational scheme that lends itself readily to the solution of a variety of radiative transfer problems in realistic planetary atmospheres.

1. INTRODUCTION

In atmospheric radiative budget calculations involving climate modeling, solar insolation modeling and prediction, air pollution studies and remote sensing, solutions of the radiative transfer equation are required.

One of the most commonly used methods for solving radiative transfer problems involving anisotropic scattering is the discrete ordinate method. Since its introduction by Chandrasekhar in the 1940's it has been utilized by many investigators to study the transfer of radiation in planetary atmospheres in the visible and the infrared regions of the spectrum [see e.g. Lenoble, 1956; Piotrowski, 1956; Samuelson, 1969; Yamamoto et al., 1971; Liou, 1973; 1974]. Although the method was originally designed to describe radiative transfer in homogeneous media, it has recently been extended for use in inhomogeneous atmospheres that can be adequately described as a series of adjacent homogeneous layers in which the scattering and absorption properties are allowed to vary from layer to layer (Liou, 1975; Liou and Sasamori, 1975; Liou et al., 1978).

The method has many appealing features which explains its frequent use.

1) Explicit expressions for the radiation field are given.

2) Useful approximate solutions (2- and 4-stream solutions) can be easily derived.

3) The computational speed is independent of the optical depth of the atmosphere.

The prime merit of the discrete ordinate method lies in its reduction of an integro-differential equation to a system of ordinary differential equations for which eigensolutions can be found. The numerical difficulties inherent in Chandresekhar's prescription for finding the eigenvalues and eigenvectors are now well known and they are briefly reviewed below (see Section 3).

The purpose of the present report is fourfold:

1) To provide a direct matrix formulation of the discrete ordinate method in which the numerical implementation is reduced to solving standard algebraic eigenvalue problems and systems of linear algebraic equations using well documented, reliable, and efficient algorithms.

2) To derive an explicit expression for the source function. This expression is used in the formal solution to the transfer equation to obtain simple analytic formulas for the intensities at <u>any</u> angle (other than just the quadrature angles).

3) To show how the method can be used to obtain integrated quantities, such as transmissivity and albedo, as a function of solar zenith angle by applying a single (isotropic) boundary condition and thus eliminate the need for a particular solution. This simplification results in substantial savings of computing time when only albedo and transmissivity are sought.

4) To provide numerical verification of the theory by comparing computed results with those obtained by other established methods. The present approach has already been used to describe the formally equivalent problem of auroral electron transport in the earth's atmosphere (cf. Stamnes, 1980).

2. THE EQUATION OF RADIATIVE TRANSFER

The equation describing the transfer of monochromatic radiation through a plane parallel homogeneous atmosphere is given by (Chandrasekhar, 1960)

$$\mu \frac{dI(\tau,\mu,\phi)}{d\tau} = I(\tau,\mu,\phi) - J(\tau,\mu,\phi) \tag{1}$$

where $I(\tau,\mu,\phi)$ is the specific intensity at level τ in a cone of unit solid angle along the direction μ,ϕ. τ is the optical depth, ϕ is the azimuthal angle and μ is the cosine of the polar angle. Unless otherwise stated, no distinction is made between direct and diffuse intensity. Thus, $I(\tau,\mu,\phi)$ is the total (diffuse + direct) intensity. The source function is given by

$$J(\tau,\mu,\phi) = \frac{\omega_0}{4\pi} \int_0^{2\pi} \int_{-1}^{1} p(\mu,\phi;\mu',\phi')\, I(\tau,\mu',\phi')\, d\mu'\, d\phi' + Q(\tau,\mu,\phi) \tag{2}$$

where the last term represents an internal source (for the moment unspecified). Note that if the usual diffuse-direct distinction is made, so that $I(\tau,\mu,\phi)$ in (1) and (2) describes the scattered intensity only, then for a parallel beam of sunlight incident on a non-emitting atmosphere

$$Q(\tau,\mu,\phi) = \frac{\omega_0 I^-(0)}{4\pi}\, p(\mu,\phi;-\mu_0,\phi_0) e^{-\tau/\mu_0} \tag{3}$$

where μ_0 is the cosine of the solar zenith angle and $\mu_0 I^-(0)$ is the incident solar flux.

Expanding the phase function, $p(\mu,\phi;\mu',\phi')$, in a series of 2n-1 Legendre polynomials using the addition theorem for spherical harmonics, one finds

$$p(\mu,\phi;\mu',\phi') =$$

$$\sum_{m=0}^{2n-1} (2-\delta_{0,m}) \sum_{\ell=m}^{2n-1} (2\ell+1)\ g_\ell^m\ P_\ell^m(\mu')P_\ell^m(\mu)\cos m(\phi'-\phi) \tag{4}$$

with

$$g_\ell^m = g_\ell \frac{(\ell-m)!}{(\ell+m)!} \quad , \quad \delta_{0,m} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$$

The coefficients g_ℓ are defined by $g_\ell =$ $1/2 \int_{-1}^{1} P_\ell(\cos\Theta)\ p(\cos\Theta)\ d(\cos\Theta)$ where $p(\cos\Theta)$ is the phase function and Θ the scattering angle which relates to μ,ϕ and μ',ϕ' through $\cos\Theta = \mu\mu' + \sqrt{1-\mu'^2}\ \sqrt{1-\mu^2}\ \cos(\phi-\phi')$.

Noting that (4) is a Fourier cosine series expansion of the phase function, one may expand the intensity and source term likewise

$$I(\tau,\mu,\phi) = \sum_{m=0}^{2n-1} I^m(\tau,\mu)\ \cos m(\phi_0-\phi) \tag{5}$$

$$Q(\tau,\mu,\phi) = \sum_{m=0}^{2n-1} Q^m(\tau,\mu)\ \cos m(\phi_0-\phi) \tag{6}$$

Substituting (4), (5), and (6) into (1) then leads to 2n independent equations

$$\mu\frac{dI^m(\tau,\mu)}{d\tau} = I^m(\tau,\mu) -$$

$$\frac{\omega_0}{2} \sum_{\ell=m}^{2n-1} (2\ell+1) g_\ell^m P_\ell^m(\mu) \int_{-1}^{1} P_\ell^m(\mu') I^m(\tau,\mu') d\mu' - Q^m(\tau,\mu) \qquad (7)$$

$m = 0,1,\ldots,2n-1$

In general (7) has to be solved subject to the boundary conditions

$$I^m(0,-\mu) = I_0^m(-\mu)$$
$$I^m(\tau^*,+\mu) = I_g^m(+\mu)$$

describing the radiation incident on the top and bottom of the atmosphere, respectively. τ^* is the optical thickness of the plane parallel atmosphere. For details on how to include the boundary conditions the reader is referred to Section 5.

Since (7) has to be solved for each m, the $m = 0$ case may be conveniently picked for the following demonstrations. It is emphasized, however, that the matrix methods to be developed in Section 4 apply equally well to $m \neq 0$.

3. CRITIQUE OF PREVIOUS NUMERICAL PROCEDURES

The discrete ordinate approximation to (7) now consists of replacing the integral by a quadrature sum so that (7) is replaced by the following system of 2n coupled first order, linear differential equations (dropping the superscript for the m = 0 case)

$$\mu_i \frac{dI_i(\tau)}{d\tau} = I_i(\tau) - \frac{1}{2} \sum_{\ell=0}^{2n-1} \tilde{\omega}_\ell P_\ell(\mu_i) \sum_{j=-n}^{n} a_j P_\ell(\mu_j)\, I_j(\tau) - Q_i(\tau) \qquad (8)$$

$$i = \pm 1, \ldots, \pm n$$

where $I_i(\tau) \equiv I(\tau,\mu_i)$, $Q_i(\tau) \equiv Q(\tau,\mu_i)$, $\tilde{\omega}_\ell \equiv \omega_0(2\ell+1)g_\ell$ and the a_j's are the quadrature weights.

Chandrasekhar (1960) has shown that the homogeneous version of (8) admits a solution of the form

$$I_i = g(\mu_i)e^{-k\tau} \qquad (9)$$

where

$$g(\mu) = \frac{1}{1+k\mu} \sum_{\ell=0}^{2n-1} \tilde{\omega}_\ell\, \xi_\ell(k)\, P_\ell(\mu) \qquad (10)$$

and the ξ_ℓ's can be obtained from the recursion relation (cf. Chandrasekhar, 1960, p. 153)

$$\xi_{\ell+1} = -\frac{2\ell+1-\tilde{\omega}_\ell}{k(\ell+1)}\, \xi_\ell - \frac{\ell}{\ell+1}\, \xi_{\ell-1}\ , \quad \ell=0,1,\ldots,2n-1 \qquad (11)$$

with $\xi_0(k) = 1$. By substituting (9) into (8) the following characteristic equation for the eigenvalues is obtained (Chandrasekhar, 1960; Liou, 1973)

$$\xi_\ell(k_i) = \frac{1}{2} \sum_{j=-n}^{n} a_j P_\ell(\mu_j) g_i(\mu_j) \qquad (12)$$

Previous applications of the discrete ordinate method (cited in the Introduction) are all based on Eqs. (10)-(12) which are the key equations in Chandrasekhar's procedure. The most comprehensive (and also the most recent) discussion of the computational problems encountered in the numerical implementation of these equations is given by Liou (1973). He found that the characteristic equation for finding the eigenvalues [Eq. (12)] had mathematical as well as numerical ambiguities and required a large amount of computer time, especially for large n. These problems led to his development of a "matrix method" for determining the eigenvalues. Asano (1975) showed that the degree of the characteristic equation for the eigenvalues can be reduced so that one may solve an n-dimensional characteristic equation for k^2 instead of a 2n-dimensional equation for k.

Both Asano (1975) and Liou (1973) solved the characteristic equation by expanding the matrix in polynomial form. As discussed in detail by Wilkinson (1965) the expansion in polynomial form is an inherently unstable procedure for obtaining eigenvalues. For further elaboration of this matter the reader is referred to the standard treatise on the subject by Wilkinson (1965). Eigenvalues and eigenvectors of a general real matrix are perhaps best obtained by an algorithm utilizing the QR transformation of Francis (1961, 1962). Standard and reliable algebraic eigenvalue routines based on such algorithms are now readily available [e.g. those implemented through the IMSL (1975) Library].

The solution of the characteristic equation adopted by Liou (1973) and Asano (1975) provides eigenvalues only. Using Chandrasekhar's procedure based on (10) and (11) to obtain eigenvectors, Liou (1973) found that for small eigenvalues, k, the ξ-values determined by (11) become inaccurate for large ℓ because of round-off errors. Thus, for small eigenvalues, Liou had to resort to lengthy stabilization schemes to determine the eigenvectors. The reason for this inaccuracy is discussed by Asano (1975) who points out that for k-values less than unity the $\xi_\ell(k)$ decrease in an oscillating manner with increasing ℓ. As a consequence the upward recursion (11) leads to a rapid loss of accuracy due to cancellations. Asano proposed using a downward recursion relation for small eigenvalues and showed that this approach is superior to the time-consuming iteration scheme suggested by Liou (1973). However, the availability of standard and reliable eigenvalue routines (based on stable algorithms) which also provide eigenvectors, renders obsolete procedures for obtaining eigenvectors using expressions like (10) and (11).

As shown by Asano (1975) it is possible to reduce the order of the eigenvalue problem by a factor of 2. In order to take full advantage of that reduction it is necessary to relate the eigenvectors of the reduced system to those of the original system (this point was not addressed by Asano). In the following section an alternative derivation of the reduced system of equations is given which explicitly reveals how the eigenvectors of the two systems are related.

4. DIRECT MATRIX SOLUTION

Assuming that the quadrature weights and points used in (8) satisfy $a_{-j} = a_j$ and $\mu_{-j} = -\mu_j$ (which is satisfied by e.g. the Gaussian quadrature of even order) the homogeneous version of (8) may be rewritten as

$$\mu_i \frac{dI_i}{d\tau} = I_i - \left\{ \sum_{j=1}^{n} C_{i,-j} I_{-j} + \sum_{j=1}^{n} C_{ij} I_j \right\} \qquad (13)$$

$$i = \pm 1, \ldots, \pm n$$

where

$$C_{ij} = \frac{\omega_0}{2} \sum_{\ell=0}^{2n-1} a_j (2\ell+1) g_\ell P_\ell(\mu_i)\, P_\ell(\mu_j) \qquad (14)$$

This system of 2n coupled differential equations can be written in matrix form as

$$\begin{bmatrix} \dfrac{d\underline{I}^+}{d\tau} \\ \\ \dfrac{d\underline{I}^-}{d\tau} \end{bmatrix} = \begin{bmatrix} -\underset{\sim}{\alpha} & -\underset{\sim}{\beta} \\ \\ \underset{\sim}{\beta} & \underset{\sim}{\alpha} \end{bmatrix} \begin{bmatrix} \underline{I}^+ \\ \\ \underline{I}^- \end{bmatrix} \; ; \quad \underline{I}^{\pm} = \begin{bmatrix} I(\tau,\pm\mu_1) \\ I(\tau,\pm\mu_2) \\ \vdots \\ I(\tau,\pm\mu_n) \end{bmatrix}$$

where the elements of the n x n matrices $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are given by

$$\alpha_{ij} \equiv \frac{1}{\mu_i}(C_{ij}-\delta_{ij}) = \frac{1}{\mu_i}(C_{-i,-j}-\delta_{ij})$$

$$\beta_{ij} \equiv \frac{1}{\mu_i} C_{-ij} = \frac{1}{\mu_i} C_{i,-j} \qquad (15)$$

Seeking solutions of the form $\underline{I}^{\pm} = \underline{g}^{\pm} e^{-k\tau}$ leads to

$$\begin{bmatrix} \underset{\sim}{\alpha} & \underset{\sim}{\beta} \\ -\underset{\sim}{\beta} & -\underset{\sim}{\alpha} \end{bmatrix} \begin{bmatrix} \underline{g}^{+} \\ \underline{g}^{-} \end{bmatrix} = k \begin{bmatrix} \underline{g}^{+} \\ \underline{g}^{-} \end{bmatrix} \tag{16}$$

which may be solved immediately as a standard eigenvalue problem. However, it is well known that the eigenvalues occur in pairs ($\pm k$) (see e.g. Chandrasekhar, 1960) and it has also been proven that they are all real (Kuscer and Vidav, 1969). The fact that the eigenvalues occur in pairs suggests that (16) can be manipulated so that one may solve for k^2 rather than k and thus reduce the order of the problem by 2. This is accomplished as follows: If one writes (16) as

$$\underset{\sim}{\alpha}\underline{g}^{+} + \underset{\sim}{\beta}\underline{g}^{-} = k\underline{g}^{+} \tag{17}$$

$$-\underset{\sim}{\beta}\underline{g}^{+} - \underset{\sim}{\alpha}\underline{g}^{-} = k\underline{g}^{-} \tag{18}$$

and then adds and subtracts (17) and (18) one finds

$$(\underset{\sim}{\alpha}-\underset{\sim}{\beta})\ (\underline{g}^{+}- \underline{g}^{-}) = k(\underline{g}^{+}+ \underline{g}^{-}) \tag{19}$$

$$(\underset{\sim}{\alpha}+\underset{\sim}{\beta})\ (\underline{g}^{+}+ \underline{g}^{-}) = k(\underline{g}^{+}- \underline{g}^{-}) \tag{20}$$

Substituting (20) into (19) yields

$$(\underset{\sim}{\alpha}-\underset{\sim}{\beta})\ (\underset{\sim}{\alpha}+\underset{\sim}{\beta})\ (\underline{g}^{+}+ \underline{g}^{-}) = k^2(\underline{g}^{+}+ \underline{g}^{-}) \tag{21}$$

This completes the reduction of the order of the algebraic eigenvalue problem and shows explicitly how the eigenvectors of the original system ($\underline{g}^{\pm}$) can be obtained from those of the reduced system ($\underline{g}^{+} \pm \underline{g}^{-}$).

The homogeneous solution can thus be written

$$I(\tau,\mu_i) = \sum_{j=-n}^{n} L_j g_j(\mu_i) e^{-k_j\tau} \tag{22}$$

where the L_j's are constants of integration to be determined from the boundary conditions.

To find a particular solution the source term must be specified. For the sake of demonstration and simplicity it is assumed that the source term can be approximated by a polynomial $Q(\tau,\mu) = \sum_{r=0}^{M} X_r(\mu)\tau^r$. This particular form is useful when treating infrared radiative transfer in non-isothermal atmospheres, and corresponds to approximating the Planck function by $B(\tau) = \sum_{r=0}^{M} X_r(\mu)\tau^r$. With this assumption the discrete ordinate approximation to (1) becomes

$$\mu_i \frac{dI_i}{d\tau} =$$

$$I_i - \frac{\omega_0}{2} \sum_{\ell=0}^{2n-1} (2\ell+1) g_\ell P_\ell(\mu_i) \sum_{j=-n}^{n} a_j P_\ell(\mu_j) I_j - \sum_{r=0}^{M} X_r(\mu_i)\tau^r \tag{23}$$

Seeking a particular solution of the form

$$I_p(\tau,\mu_i) = \sum_{r=0}^{M} Z_r(\mu_i)\tau^r \tag{24}$$

and substituting (24) into (23) and equating coefficients of like powers of τ, one obtains

$$\sum_{j=-n}^{n} (\delta_{ij} - C_{ij}) Z_M(\mu_i) = X_M(\mu_i)$$

$$\sum_{j=n}^{n} (\delta_{ij} - C_{ij}) Z_r(\mu_i) = X_r(\mu_i) - (r+1)\mu_i Z_{r+1}(\mu_i) \tag{25}$$

$r = M-1, M-2, \ldots, 0.$

where C_{ij} is given by (14). Noting that (25) is a system of linear algebraic equations determining the $Z_r(\mu_i)$'s the general solution can now be written

$$I(\tau,\mu_i) = \sum_{j=-n}^{n} L_j g_j(\mu_i) e^{-k_j\tau} + \sum_{r=0}^{M} Z_r(\mu_i)\tau^r \tag{26}$$

The solution is completed by specifying the boundary conditions which leads to yet another system of linear algebraic equations for determining the L_j's (cf. Section 5).

Thus, it is seen that the present approach to the discrete ordinate method can be numerically implemented by using standard (but highly sophisticated) eigenvalue routines and linear algebraic equation solvers. Using these methods, the numerical problems which Liou runs into are avoided and it can therefore be concluded that these problems are not inherent in the method but rather results from using inadequate numerical procedures. The present approach is computationally faster than that of Liou and others, partly due to the reduction of the order of the problem by a factor of 2, and partly due to the faster calculation of eigenvalues and eigenvectors.

5. BOUNDARY CONDITIONS

The purpose of this section is to discuss the inclusion of the boundary conditions needed to complete the solution. In principle, this amounts to specifying the radiation incident at the top and bottom of the atmosphere. The discussion is restricted to the azimuth independent case for which only the zeroth order Fourier-component contributes. The higher order components can be treated quite analogously. For the following demonstrations we assume that the atmosphere is illuminated from above by a known source of radiation and that the underlying ground is a non-emitting Lambert reflector with albedo A. For these boundary conditions we have

$$I(\tau = 0, -\mu) = I^{\infty}(-\mu) \qquad \text{(known source)} \qquad (27)$$

$$0 \leq \mu \leq 1$$

$$I(\tau = \tau^*, +\mu) = \frac{A}{\pi} \int_0^{2\pi} d\phi \int_0^1 d\mu \mu I(\tau^*, -\mu) \qquad (28)$$

where τ^* is the total optical thickness of the atmosphere. While (27) describes the known radiation incident on the top of the atmosphere, (28) describes the reflection by the underlying surface. For a Lambert reflector the reflected intensity is assumed to be unpolarized and isotropic regardless what the state of the incident radiation might be.

Since (27) and (28) introduce a fundamental distinction between downward directions (denoted by $-\mu$) and upward directions (denoted by $+\mu$), it would be beneficial to select a quadrature rule which integrates separately over the downward and upward directions. In contrast to a quadrature rule that is selected to integrate only the complete range $-1<\mu<1$, such a choice also has the advantage that no further approximations are needed to obtain upward and downward fluxes [see e.g. Eq. (28)]. The quadrature rule adopted by Chandrasekhar (cf. Chandrasekhar, 1960) and also used by several subsequent investigators [e.g. Lenoble (1956), Samuelson (1969), Yamamoto et al. (1971), Liou (1973)] is the Gaussian formula for the complete range $-1<\mu<1$. However, if the Gaussian formula is applied separately to the half ranges $-1<\mu<0$ and $0<\mu<1$ (Sykes, 1951) a quadrature rule is obtained which directly incorporates the fundamental distinction between downward and upward directions into the formalism. This quadrature formula which Sykes (1951) referred to as "double-Gauss" is adopted in the present work. The advantages of this choice are discussed at some length in a paper by Krook (1955).

In the following we first discuss the inclusion of the boundary conditions in quite general terms and then we show what simplifications are possible for the case of a parallel incident beam and in particular for calculations in which only albedo and transmissivity are required.

a. General

The discrete ordinate approximation to (27) describing the radiation field incident on the top of the atmosphere is

$$I(0, -\mu_i) = I_i^{\infty} \quad ; \; i = 1,\ldots,n \tag{29}$$

where I_i^{∞} is the amount of radiation incident in direction $\theta_i = \text{arc}\cos\mu_i$. Using (26) and (24) we may rewrite (29) as

$$\sum_{j=1}^{n} \left\{ L_j g_j(\mu_i) + L_{-j} g_{-j}(-\mu_i) \right\} = I_i^{\infty} - I_p(0,-\mu_i) \qquad (30)$$

$$; \quad i = 1,\ldots,n$$

At the bottom of the atmosphere we require

$$\pi I_i(\tau^*) = A \quad 2\pi \sum_{s=1}^{n} \quad a_s \mu_s I(\tau^*,-\mu_s) \qquad (31)$$

$$; \quad i = 1,\ldots,n$$

which simply states that the flux reflected from the underlying ground (assuming that the reflected intensity is isotropic) is proportional to the incident flux. Here the a_s's and the μ_s's are the appropriate "double-Gaussian" quadrature weights and divisions, respectively. Equation (31) may be written in the form

$$\sum_{j=1}^{n} \left\{ L_j \; g_j(\mu_i) \; r_{ij} \; e^{-k_j \tau^*} + L_{-j} \; g_{-j}(\mu_i) \; r_{i,-j} \; e^{k_j \tau^*} \right\}$$

$$= S_i(\tau^*) \quad ; \quad i = 1,\ldots,n \qquad (32)$$

where

$$r_{ij} \equiv 1 - 2A \sum_{s=1}^{n} \; a_s \; \mu_s \frac{g_j(-\mu_s)}{g_j(\mu_i)} \qquad (33)$$

and

$$S_i(\tau^*) \equiv 2A \sum_{s=1}^{n} a_s \mu_s I_p(\tau^*,-\mu_s) - I_p(\tau^*,\mu_i) \qquad (34)$$

$$; \quad i = 1,\ldots,n$$

Equations (30) and (32) are 2n equations from which the 2n unknowns, the L_j's, are to be determined. However, as pointed out by Samuelson (1969) these equations are ill-conditioned for optical depths $\tau^*>1$. This ill-conditioning is due to the great disparity in magnitude of the coefficients $\exp(\pm k_j\tau^*)$ in (32). Yamamoto et al. (1971) showed that the ill-conditioning can be avoided by solving for

$$\left.\begin{aligned} M_j &= L_j + L_{-j}\, e^{k_j\tau^*} \\ N_j &= L_j - L_{-j}\, e^{k_j\tau^*} \end{aligned}\right\} \quad ; \; j = 1,\ldots,n \qquad (35)$$

and then obtain the L_j's from

$$\left.\begin{aligned} L_j &= 1/2(M_j + N_j) \\ L_{-j} &= 1/2\,(M_j - N_j)e^{-k_j\tau^*} \end{aligned}\right\} \quad ; \; j = 1,\ldots,n \qquad (36)$$

Thus, by adding and then subtracting (30) and (32) the resulting equations can be written in matrix form as

$$\begin{bmatrix} \underset{\sim}{a} + \underset{\sim}{b} & -(\underset{\sim}{c}-\underset{\sim}{d}) \\ \underset{\sim}{c} + \underset{\sim}{d} & -(\underset{\sim}{a}-\underset{\sim}{b}) \end{bmatrix} \begin{bmatrix} \underset{\sim}{M} \\ \underset{\sim}{N} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{T} \\ \underset{\sim}{B} \end{bmatrix} \qquad (37)$$

where

$$
\left.\begin{aligned}
T_i &= I_i^{\infty} - I_p(0,-\mu_i) + S_i(\tau^*) \\
B_i &= I_i^{\infty} - I_p(0,-\mu_i) - S_i(\tau^*)
\end{aligned}\right\} \tag{38a}
$$

and

$$
\left.\begin{aligned}
a_{ij} &= 1/2(1 + r_{ij})g_j(\mu_i)e^{-k_j\tau^*} \\
b_{ij} &= 1/2(1 + r_{i,-j})g_{-j}(\mu_i) \\
c_{ij} &= 1/2(1 - r_{ij})g_j(\mu_i)e^{-k_j\tau^*} \\
d_{ij} &= 1/2(1 - r_{i,-j})g_{-j}(\mu_i)
\end{aligned}\right\} \tag{38b}
$$

and M_j and N_j are defined in (35). In (37) the column vectors $\underset{\sim}{M}$, $\underset{\sim}{N}$, $\underset{\sim}{T}$ and $\underset{\sim}{B}$ are of dimension n while the matrices $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, and $\underset{\sim}{d}$ are of dimension n x n. Hence, (37) is a system of 2n algebraic equations. For the standard problem of a perfectly absorbing ground (A = 0) it follows from (33) that $r_{ij} = 1$ implying [(38b)] $\underset{\sim}{c} = \underset{\sim}{d} = \underset{\sim}{0}$. Thus, in this case the problem is reduced to solving the following two systems of n equations instead of the original system of order 2n

$$
\left.\begin{aligned}
(\underset{\sim}{a} + \underset{\sim}{b})\underset{\sim}{M} &= \underset{\sim}{T} \\
(\underset{\sim}{a} - \underset{\sim}{b})\underset{\sim}{N} &= \underset{\sim}{B}
\end{aligned}\right\} \tag{39}
$$

where $a_{ij} = g_j(\mu_i)e^{-k_j\tau^*}$ and $b_{ij} = g_{-j}(\mu_i)$.

b. Parallel incident beam

As noted in Section 2 if the usual diffuse-direct distinction is made, so that $I(\tau,\mu_i)$ in (3) describes the scattered (or diffuse) radiation only, then for a parallel beam of sunlight incident on a non-emitting atmosphere the particular solution becomes

$$I_p(\tau,\mu_i) = Z_0(\mu_i)e^{-\tau/\mu_0} \quad (40)$$

where μ_0 is the cosine of the solar zenith angle. The $Z_0(\mu_i)$'s are readily determined from the following system of linear algebraic equations

$$\sum_{j=-n}^{n} [1 + \mu_i/\mu_0)\delta_{ij} - C_{ij}]Z_0(\mu_j) = X_0(\mu_i) \quad (41)$$

where

$$X_0(\mu_i) = \frac{\omega_0 I^-(0)}{4\pi} \sum_{\ell=0}^{2n-1} (-1)^{\ell} P_\ell(\mu_i) P_\ell(\mu_0) \quad (42)$$

The boundary conditions pertinent to the case of a parallel incident beam are

$$I(0,-\mu_i) = 0 \quad (43)$$

$$I(\tau^*,+\mu_i) = \frac{A}{\pi}\left\{ 2\pi \sum_{s=1}^{n} a_s\mu_s \, I(\tau^*,-\mu_s) + \mu_0 I^-(0)e^{-\tau^*/\mu_0} \right\} \quad (44)$$

Equation (43) describes the fact that no diffuse radiation is incident on the top of the atmosphere whereas in (44) the direct attenuated beam is included to describe properly the Lambertian reflection by the underlying surface.

It follows readily that the constants of integration, the L_j's, can again be obtained from (37) and (38) with $I_i^{\infty} = 0$ (no incident radiation) and $I_p(\tau,\mu_i)$ given by (40).

c. Reflection and transmission

As discussed in Section 7 substantial computational shortcuts are possible if only reflected and transmitted fluxes are required. In this case only the standard problem of a nonreflecting ground (A = 0) needs consideration since the solution to the planetary problem (A ≠ 0) follows immediately form the solution to the standard problem.

Further simplification is feasible because the albedo and transmissivity for a parallel incident beam are proportional to the angular distributions of the reflected and transmitted intensities, respectively, resulting from an isotropic incident beam (cf. Section 7). This implies that there is no need for a particular solution in this case and only one (isotropic) boundary condition is required to obtain albedo and transmissivity for any direction of the incident parallel beam. Consequently, the proper boundary conditions are

$$I(0,-\mu_i) = I^{-}(0) \qquad (45)$$

$$i = 1,\ldots,n$$

$$I(\tau^{*},+\mu_i) = 0 \qquad (46)$$

The constants of integration are readily obtained from the system of equations (39) with $T_i = B_i = I^{-}(0)$ appropriate for the standard problem with A = 0 (black ground).

6. SOURCE FUNCTION AND ANGULAR DISTRIBUTIONS

The purpose of this section is to show that an explicit expression for the source function can be derived by the discrete ordinate method. This expression is used in the formal solution to the transfer equation to obtain new expressions for the angular distributions from which the intensity at <u>any</u> angle can be estimated. This method was used by Devaux et al. (1973), Dave and Armstrong (1974) and Dave (1975) to obtain smooth angular distributions by the spherical harmonic method.

The formal solution to (1) is easily obtained as (Chandrasekhar, 1960)

$$I(\tau,+\mu) = I(\tau^*,+\mu)e^{-(\tau^*-\tau)/\mu} + \int_{\tau}^{\tau^*} J(\tau,+\mu)e^{-(t-\tau)/\mu}\,\frac{dt}{\mu}$$

$$I(\tau,-\mu) = I(0,-\mu)e^{-\tau/\mu} + \int_{0}^{\tau} J(\tau,-\mu)e^{-(\tau-t)/\mu}\,\frac{dt}{\mu} \qquad (47)$$

where the source function is given by (2) (for the azimuth independent case)

$$J(\tau,\mu) = \frac{\omega_0}{2}\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' + Q(\tau,\mu) \qquad (48)$$

a. Homogeneous case

Considering first the case $Q(\tau,\mu) = 0$ it follows that the discrete ordinate approximation to (48) is [see Eq. (8)]

$$J(\tau,\mu) = \frac{1}{2}\sum_{\ell=0}^{2n-1} \tilde{\omega}_\ell P_\ell(\mu) \sum_{j=-n}^{n} P_\ell(\mu_j)a_j I(\tau,\mu_j) \qquad (49)$$

Using (22) one may rewrite (49) as

$$J(\tau,\mu) =$$

$$\frac{1}{2} \sum_{\ell=0}^{2n-1} \tilde{\omega}_\ell P_\ell(\mu) \sum_{j=-n}^{n} P_\ell(\mu_j) a_j \sum_{i=-n}^{n} L_i g_i(\mu_j) e^{-k_i \tau} \tag{50}$$

By rearranging sums and using (10) and (12), (50) is reduced to

$$J(\tau,\mu) = \sum_{j=-n}^{n} (1+k_j\mu) L_j g_j(\mu) e^{-k_j \tau} \tag{51}$$

Using this expression in the formal solution (47) yields

$$I(\tau,+\mu) = I(\tau^*,+\mu) e^{-(\tau^*-\tau)/\mu} + \sum_{j=-n}^{n} L_j g_j(\mu) \left\{ e^{-k_j\tau} - e^{-[k_j\tau^* + \frac{1}{\mu}(\tau^*-\tau)]} \right\} \tag{52}$$

$$I(\tau,-\mu) = I(0,-\mu) e^{-\tau/\mu} + \sum_{j=-n}^{n} L_j g_j(-\mu) \left\{ e^{-k_j\tau} - e^{-\tau/\mu} \right\} \tag{53}$$

These expressions can be used to estimate angular distributions in any order of approximation. The $g_i(\mu)$'s can be determined by the analytic interpolation formula provided by (10) and (12).

b. Inhomogeneous case

Considering next the case $Q(\tau,\mu) \neq 0$ and assuming, as before, that one can write $Q(\tau,\mu) = \sum_{r=0}^{M} X_r(\mu) \tau^r$ the source function becomes

$$J(\tau,\mu) = \frac{1}{2} \sum_{\ell=0}^{2n-1} \tilde{\omega}_\ell P_\ell(\mu) \int_{-1}^{1} P_\ell(\mu') I(\tau,\mu') d\mu' + \sum_{r=0}^{M} X_r(\mu) \tau^r \tag{54}$$

Using (26) this equation becomes

$$J(\tau,\mu) = \frac{1}{2} \sum_{\ell=0}^{2n-1} \tilde{\omega}_\ell P_\ell(\mu) \sum_{j=-n}^{n} a_j P_\ell(\mu_j) \sum_{i=-n}^{n} L_i g_i(\mu_j) e^{-k_i\tau} + \sum_{r=0}^{M} \left\{ Z_r(\mu_j) \sum_{\ell=0}^{2n-1} \frac{1}{2}\tilde{\omega}_\ell P_\ell(\mu) \sum_{j=-n}^{n} a_j P_\ell(\mu_j) + X_r(\mu) \right\} \tau^r \tag{55}$$

Since $I_p(\tau,\mu) = \sum_{r=0}^{M} Z_r(\mu)\tau^r$ is a particular solution to the transfer equation it follows that

$$J(\tau,\mu) = \sum_{j=-n}^{n} (1+k_j\mu) L_j g_j(\mu) e^{-k_j\tau} + \sum_{r=0}^{M} [Z_r(\mu) - \mu(r+1) Z_{r+1}(\mu)] \tau^r \tag{56}$$

The angular distributions can now be written down immediately as

$$I(\tau,+\mu) = \sum_{j=-n}^{n} L_j g_j(\mu) \left\{ e^{-k_j\tau} - e^{-[k_j\tau^* + \frac{1}{\mu}(\tau^*-\tau)]} \right\}$$

$$+ \sum_{r=0}^{M} [Z_r(\mu)-\mu(r+1)Z_{r+1}(\mu)]F_r(\tau,\mu) \tag{57}$$

$$+ I(\tau^*,+\mu)e^{-(\tau^*-\tau)/\mu}$$

where

$$F_r(\tau,\mu) = \sum_{p=0}^{r} \frac{r!\mu^p}{(r-p)!} \left\{ \tau^{r-p} - \tau^{*r-p} e^{-(\tau^*-\tau)/\mu} \right\}$$

$$I(\tau,-\mu) = I(0,-\mu)e^{-\tau/\mu} + \sum_{j=-n}^{n} L_j g_j(-\mu) \left\{ e^{-k_j\tau} - e^{-\tau/\mu} \right\}$$

$$+ \sum_{r=0}^{M} [Z_r(-\mu) + \mu(r+1)Z_{r+1}(-\mu)]G_r(\tau,\mu) \tag{58}$$

where $G_r(\tau,\mu) = \sum_{p=0}^{r} \frac{(-1)^p \mu^p r! \tau^{r-p}}{(r-p)!} - (-1)^r \mu^r r! e^{-\tau/\mu}$

Equation (25) determines the $Z_r(\mu_j)$'s at the quadrature points and that information can be used to evaluate the Z_r's at any angle by interpolation.

c. Parallel incident beam

The discrete ordinate approximation to the source function for this case is given by [see Eq. (3)]

$$J(\tau,\mu) = \frac{1}{2} \sum_{\ell=0}^{2n-1} \tilde{\omega}_\ell P_\ell(\mu) \sum_{j=-n}^{n} a_j P_\ell(\mu_j) I(\tau,\mu_j)$$

$$+ \frac{\omega_0 I^-(0)}{4\pi} \sum_{\ell=0}^{2n-1} (-1)^\ell P_\ell(\mu) P_\ell(\mu_0) e^{-\tau/\mu_0} \tag{59}$$

Proceeding as in the previous case one finds that the source function and the angular distribution of the diffuse intensities can be expressed as

$$J(\tau,\mu) = \sum_{j=-n}^{n} (1+k_j\mu) L_j g_j(\mu) e^{-k_j\tau}$$

$$+ (1 + \mu/\mu_0) Z_0(\mu) e^{-\tau/\mu_0} \tag{60}$$

$$I(\tau,+\mu) = \sum_{j=-n}^{n} L_j g_j(\mu) \left\{ e^{-k_j\tau} - e^{-[k_j\tau^* + \frac{1}{\mu}(\tau^*-\tau)]} \right\}$$

$$+ Z_0(\mu) \left\{ e^{-\tau/\mu_0} - e^{-[\tau^*/\mu_0 + \frac{1}{\mu}(\tau^*-\tau)]} \right\} \tag{61}$$

$$I(\tau,-\mu) = \sum_{j=-n}^{n} L_j g_j(-\mu) \left\{ e^{-k_j\tau} - e^{-\tau/\mu} \right\}$$

$$+ Z_0(-\mu) \left\{ e^{-\tau/\mu_0} - e^{-\tau/\mu} \right\} \tag{62}$$

Although the $Z_0(\mu)$'s can be estimated using standard interpolation techniques, the following formula (which can be derived directly from the basic equations) is useful

$$Z_0(\mu) = \frac{1}{1+\mu/\mu_0} \frac{1}{2} \sum_{\ell=0}^{2n-1} P_\ell(\mu) \left\{ \tilde{\omega}_\ell \sum_{j=-n}^{n} a_j P_\ell(\mu_j) Z_0(\mu_j) + \frac{\omega_0 I^-(0)}{2\pi} (-1)^\ell P_\ell(\mu_0) \right\} \quad (63)$$

7. SIMPLIFIED REFLECTION AND TRANSMISSION CALCULATIONS

As a consequence of the reciprocity principle certain integrated quantities have a dual interpretation. In this section it is shown that this duality can be used to make substantial computational shortcuts if only albedo and transmissivity are required.

It is convenient to distinguish between the standard problem in which there is no ground reflection and the planetary problem in which there is a reflecting ground at the bottom of the atmosphere. This distinction is useful because it has been shown (Chandrasekhar, 1960) that the solution of the planetary problem can be reduced to that of the standard problem.

a. The standard problem

Consider a parallel beam of light incident on a plane parallel atmosphere with intensity

$$I^-(0,-\mu) = I^-(0)\delta(\mu-\mu_0)\delta(\phi-\phi_0) \quad (64)$$

The azimuth independent diffuse intensity emerging from the top of the atmosphere can then be expressed as (see e.g. Chandrasekhar, 1960, p. 20).

$$I(0,\mu,\mu_0) = \frac{I^-(0)}{4\pi\mu} S(\tau^*,\mu,\mu_0) \quad (65)$$

where $S(\tau^*,\mu,\mu_0)$ is referred to as the scattering function for a plane parallel slab of thickness τ^*. Similarly, the intensity diffusely transmitted by the atmosphere is

$$I(\tau^*,-\mu,\mu_0) = \frac{I^-(0)}{4\pi\mu} T(\tau^*,\mu,\mu_0) \qquad (66)$$

where $T(\tau^*,\mu,\mu_0)$ is called the transmission function. It is well known that the scattering and transmission functions are symmetric in μ and μ_0 and this symmetry property is what is known as the principle of reciprocity as it applies to the problem of diffuse reflection and transmission by a plane parallel atmosphere (Chandrasekhar, 1960, p. 172).

The angular distribution of the intensity diffusely reflected and transmitted by the atmosphere for an <u>isotropic</u> incident illumination is obtained from (65) and (66), respectively, as

$$I(0,\mu) = 2\pi \int_0^1 I(0,\mu,\mu_0)d\mu_0 = \frac{I^-(0)}{2\mu} \int_0^1 S(\tau^*,\mu,\mu_0)d\mu_0 \qquad (67)$$

$$I(\tau^*,-\mu) = 2\pi \int_0^1 I(\tau^*,-\mu,\mu_0)d\mu_0 = \frac{I^-(0)}{2\mu} \int_0^1 T(\tau^*,\mu,\mu_0)d\mu_0 \qquad (68)$$

From (67) and (68) the reflected and transmitted fluxes are found

$$F^+(0) = 2F^-(0) \int_0^1 I(0,\mu)\mu d\mu = F^-(0) \int_0^1 d\mu \int_0^1 d\mu_0\, S(\tau^*,\mu,\mu_0) \qquad (69)$$

$$F^-(\tau^*) = 2F^-(0)\int_0^1 I(\tau^*,-\mu)\mu d\mu = F^-(0)\int_0^1 d\mu\int_0^1 d\mu_0\, T(\tau^*,\mu,\mu_0) \quad (70)$$

where $F^-(0) = \pi I^-(0)$ is the incident flux corresponding to isotropic incident illumination, $I^-(0)$.

Now, the plane albedo, $a(\mu_0)$, and the transmissivity, $t(\mu_0)$, resulting from a parallel beam of light incident on the atmosphere can also be obtained from (65) and (66). Thus,

$$a(\mu_0) = \frac{2\pi}{\mu_0 I^-(0)}\int_0^1 I(0,\mu,\mu_0)\mu d\mu = \frac{1}{2\mu_0}\int_0^1 S(\tau^*,\mu,\mu_0)d\mu \quad (71)$$

$$t(\mu_0) = \frac{2\pi}{\mu_0 I^-(0)}\int_0^1 I(\tau^*,-\mu,\mu_0)\mu d\mu = \frac{1}{2\mu_0}\int_0^1 S(\tau^*,\mu,\mu_0)d\mu \quad (72)$$

and from (71) and (72) the spherical albedo and the spherical transmissivity become

$$a_s = 2\int_0^1 a(\mu_0)\mu_0 d\mu_0 = \int_0^1 d\mu\int_0^1 d\mu_0\, S(\tau^*,\mu,\mu_0) \quad (73)$$

$$t_s = 2\int_0^1 t(\mu_0)\mu_0 d\mu_0 = \int_0^1 d\mu\int_0^1 d\mu_0\, T(\tau^*,\mu,\mu_0) \quad (74)$$

Since S and T are symmetric in μ and μ_0 the results shown in Eqs. (67)-(74) may be summarized as follows:

1) The angular distribution of the emergent intensity due to isotropic incident illumination is proportional to the plane albedo resulting from a parallel incident beam, i.e.

$$a(\mu) = I(0,\mu)/I^{-}(0) \tag{75}$$

2) The angular distribution of the diffusely transmitted intensity due to isotropic incident illumination is proportional to the transmissivity caused by a parallel incident beam, i.e.

$$t(\mu) = I(\tau^{*},-\mu)/I^{-}(0) \tag{76}$$

3) The reflected flux caused by isotropic incident illumination is proportional to the spherical albedo for a parallel incident beam, i.e.

$$a_s = F^{+}(0)/F^{-}(0) \tag{77}$$

4) The transmitted flux resulting from isotropic incident illumination is proportional to the spherical transmissivity for a parallel incident beam, i.e.

$$t_s = F^{-}(\tau^{*})/F^{-}(0) \tag{78}$$

From a practical computational point of view the duality noted in 1)–4) above has an important consequence. Suppose one is interested in computing the plane albedo $a(\mu_0)$ for various values of μ_0. This calculation would have to be repeated for each value of μ_0, since changing μ_0 corresponds to changing the boundary condition. In addition, for each μ_0 a particular solution has to be found. It is much more expedient to calculate the emergent angular distribution of the diffuse intensity for an isotropic incident illumination (which is proportional to the plane albedo) since this problem is solved by applying only one boundary condition and no particular solution is required.

It follows from the above that the local albedo and the diffuse transmissivity can be calculated from

$$a(\mu_0) = \sum_{j=-n}^{n} L_j g_j(\mu_0) \left\{ 1 - e^{-\tau^*(k_j + \frac{1}{\mu_0})} \right\} \tag{79}$$

$$t(\mu_0) = \sum_{j=-n}^{n} L_j g_j(-\mu_0) \left\{ e^{-k_j \tau^*} - e^{-\tau^*/\mu_0} \right\} \tag{80}$$

where the L_j's are determined by applying an isotropic boundary condition. It is demonstrated in the following section that very accurate albedo and transmissivity values are obtained in low order approximations when these expressions are used.

b. The planetary problem

The presence of a reflecting ground at the bottom of the atmosphere will change both the plane albedo and the transmissivity. However, Chandrasekhar has shown that the solution to this problem can be expressed in terms of the solution to the standard problem. Thus, for a Lambert reflecting ground with albedo A, the law of diffuse reflection and transmission becomes (Chandrasekhar, 1960, p. 273)

$$I_A(0,\mu,\mu_0) = \frac{I^-(0)}{4\pi\mu} S_A(\tau^*,\mu,\mu_0) \tag{81}$$

$$I_A(\tau^*,-\mu,\mu_0) = \frac{I^-(0)}{4\pi\mu} T_A(\tau^*,\mu,\mu_0) \tag{82}$$

where the subscript A is used to distinguish the solution of the planetary problem from the standard problem. But the scattering and transmission functions for the planetary problem, S_A and T_A, can be expressed in terms of the solution to the standard problem as

$$S_A(\tau^*,\mu,\mu_0) = S(\tau^*,\mu,\mu_0) + \frac{4A}{1-Aa_s} \mu\mu_0 \, \gamma(\mu_0)\gamma(\mu) \tag{83}$$

$$T_A(\tau^*,\mu,\mu_0) = T(\tau^*,\mu,\mu_0) + \frac{4A}{1-Aa_s} \mu\mu_0 \, \gamma(\mu_0)a(\mu) \tag{84}$$

where $S(\tau^*,\mu,\mu_0)$ and $T(\tau^*,\mu,\mu_0)$ are defined in Eqs. (65) and (66). $a(\mu)$ is the plane albedo for the standard problem and $a_s = 2\int_0^1 a(\mu)\mu d\mu$ is the spherical albedo. $\gamma(\mu)$ is the total transmissivity, i.e. $\gamma(\mu) = e^{-\tau^*/\mu} + t(\mu)$, where $t(\mu)$ is the diffuse transmissivity for the standard problem. It follows readily that the plane albedo and the transmissivity for the planetary problem can be expressed as

$$a_A(\mu_0) = a(\mu_0) + \frac{A\gamma_s}{1-Aa_s} \gamma(\mu_0) \tag{85}$$

$$t_A(\mu_0) = t(\mu_0) + \frac{Aa_s}{1-Aa_s} \gamma(\mu_0) \tag{86}$$

where $\gamma_s = 2 \int_0^1 \gamma(\mu)\mu d\mu$ is the total spherical transmissivity. Thus it is seen that once the standard problem is solved, the solution to the planetary problem follows immediately.

For completeness it is noted that while $S_A(\tau^*,\mu,\mu_0)$ is symmetric in μ and μ_0 as required by the reciprocity principle, $T_A(\tau^*,\mu,\mu_0)$ is not. As a consequence only duality properties 1), 2) and 4) noted above for the standard problem hold true also for the planetary problem. There is no longer an equivalence between the angular distribution of the transmitted diffuse intensity, $I_A(\tau^*,-\mu)$, for an isotropic incident illumination and the transmissivity for a parallel incident beam, $T_A(\mu)$. In fact, these quantities are now related through

$$T_A(\mu) = I_A(\tau^*,-\mu) + \frac{A}{1-Aa_s} \gamma(\mu)a_s - a(\mu)\gamma_s \qquad (87)$$

8. RESULTS AND COMPARISONS

Since our considerations in this report are essentially methodological, we feel that comparisons with previously published results are essential. Unfortunately, there is a paucity of tabulated results available in the open literature. This makes a complete and exhaustive comparison with previous work difficult. We have found, however, that the situation is somewhat improved as far as albedo and transmissivity computations are concerned. Thus, we focus on this type of comparison.

a. Comparison with Liou's (1973) computations

In Tables 1 and 2 we compare results of the present discrete ordinate computations of albedo and transmissivity with those of Liou (1973) for

Table 1

Plane albedo comparison for single scattering albedo $\omega = 0.8$ and Henyey-Greenstein phase function with asymmetry factor $g = 0.75$

τ	# of streams	Liou (1973) μ_0			Present Results μ_0		
		0.1	0.5	0.9	0.1	0.5	0.9
0.25	4	0.30269	0.04040	0.01746	0.31287	0.05067	0.01641
	8	0.29599	0.04799	0.01473	0.29450	0.04814	0.01488
	16	0.29406	0.04888	0.01558	0.28963	0.04854	0.01549
	D	0.28961	0.04855	0.01547	0.28961	0.04855	0.01547
1	4	0.37646	0.12003	0.05425	0.38393	0.12637	0.05099
	8	0.36938	0.12471	0.04901	0.35815	0.12310	0.04856
	16	0.36071	0.12396	0.04942	0.35491	0.12340	0.04929
	D	0.35487	0.12342	0.04929	0.35487	0.12342	0.04929
4	4	0.39835	0.16792	0.09563	0.39803	0.16364	0.09178
	8	0.38582	0.16755	0.08907	0.37496	0.16583	0.08858
	16	0.37725	0.16677	0.08944	0.37153	0.16613	0.08925
	D	0.37148	0.16615	0.08925	0.37148	0.16615	0.08925
16	4	0.39914	0.16985	0.09945	0.39888	0.16562	0.09553
	8	0.38661	0.16947	0.09280	0.37576	0.16776	0.09228
	16	0.37805	0.16870	0.09316	0.37233	0.16806	0.09297
	D	0.37229	0.16808	0.09297	0.37229	0.16808	0.09297

Table 2

Transmissivity comparison for single scattering albedo $\omega = 0.8$ and Henyey-Greenstein phase function with asymmetry factor $g = 0.75$

τ	# of streams	Liou (1973) μ_0			Present Results μ_0		
		0.1	0.5	0.9	0.1	0.5	0.9
	4	0.46032	0.86090	0.92623	0.43113	0.84486	0.92743
	8	0.44354	0.84949	0.92728	0.42675	0.84797	0.92685
0.25	16	0.43120	0.84795	0.92679	0.43022	0.84755	0.92669
	D	0.43017	0.84756	0.92669	0.43017	0.84756	0.92669
	4	0.22724	0.52936	0.72003	0.19620	0.50059	0.72169
	8	0.20192	0.51471	0.71702	0.20553	0.51611	0.71761
1	16	0.20416	0.51601	0.71784	0.20555	0.51605	0.71773
	D	0.20556	0.51606	0.71772	0.20556	0.51606	0.71772
	4	0.04412	0.10648	0.22130	0.04815	0.11307	0.21565
	8	0.04453	0.10625	0.21918	0.04530	0.10727	0.20303
4	16	0.04505	0.10710	0.21959	0.04564	0.10720	0.21954
	D	0.04539	0.10718	0.21953	0.04539	0.10718	0.21953
	4	0.00026	0.00063	0.00138	0.00026	0.00061	0.00118
	8	0.00026	0.00062	0.00138	0.00028	0.00062	0.00139
16	16	0.00027	0.00062	0.00139	0.00028	0.00062	0.00139
	D	0.00027	0.00062	0.00139	0.00027	0.00062	0.00139

single scattering albedo $\omega = 0.8$ and Henyey-Greenstein phase function with asymmetry factor, $g = 0.75$. Liou also cites results obtained by van de Hulst and Grossman using a doubling method which is accurate to about 5 decimal places (cf. van de Hulst and Grossman, 1968). For reference and comparison these values are given in the rows marked "D" in the tables.

The superiority of the present results as compared to those of Liou (1973) is perhaps most strikingly illustrated by noting that the present 8-stream results are generally closer to the accurate doubling results than are Liou's 16-stream values. Furthermore, it is seen that 16-stream results are in most instances identical to those obtained by the doubling method of van de Hulst and Grossman. The prime reason for the excellent accuracy obtained is that our matrix formulation allows us to use standard, well documented and reliable algebraic eigenvalue routines and linear algebraic equation solvers for numerical implementation of the theory.

A similar comparison is provided in Tables 3 and 4 for the same phase function but for conservative scattering (i.e. $\omega = 1$). For $\omega = 1$ the characteristic equation has a double root $k^2 = 0$. This implies that the exponential solutions corresponding to these roots should be replaced by a solution that is proportional to the depth variable, τ. Although this can be easily done, it would be convenient not having to treat the conservative case separately. This can be practically accomplished by replacing the real constant $\omega = 1$ by $1-\varepsilon$ where ε is a small number ($\varepsilon = 10^{-n}$ if the computer uses n digits). Using this "trick" the computer returns a smallest root k^2 that instead of being identically zero is

Table 3

Plane albedo comparison for conservative scattering (ω = 1.0) and Henyey-Greenstein phase function with asymmetry factor g = 0.75

τ	# of streams	Liou (1973) μ_0			Present Results μ_0		
		0.1	0.5	0.9	0.1	0.5	0.9
0.25	4	0.40339	0.05743	0.03221	0.38543	0.07565	0.02191
	8	0.40985	0.06937	0.02114	0.41476	0.07192	0.02256
	16	0.41768	0.07165	0.02246	0.41600	0.07180	0.02249
	D	0.41610	0.07179	0.02250	0.41610	0.07179	0.02250
1	4	0.56631	0.22498	0.09826	0.58329	0.24725	0.09703
	8	0.58967	0.24037	0.09582	0.58013	0.24037	0.09672
	16	0.58567	0.24068	0.09654	0.58152	0.24048	0.09672
	D	0.58148	0.24048	0.09672	0.58148	0.24048	0.09672
4	4	0.73722	0.52120	0.34962	0.72859	0.52118	0.34883
	8	0.73877	0.52046	0.34806	0.73192	0.51918	0.34822
	16	0.73541	0.51977	0.34776	0.73257	0.51932	0.34823
	D	0.73254	0.51932	0.34823	0.73254	0.51932	0.34823
16	4	0.88407	0.78881	0.71005	0.87854	0.78615	0.70718
	8	0.88397	0.78659	0.70755	0.88074	0.78654	0.70718
	16	0.88240	0.78702	0.70627	0.88104	0.78658	0.70721
	D	0.88103	0.78659	0.70722	0.88103	0.78659	0.70722

Table 4

Transmissivity comparison for conservative scattering (ω = 1.0) and Henyey-Greenstein phase function with asymmetry factor g = 0.75

τ	# of streams	Liou (1973) μ_0			Present Results μ_0		
		0.1	0.5	0.9	0.1	0.5	0.9
	4	0.59661	0.94257	0.97593	0.61456	0.92435	0.97808
	8	0.59015	0.93063	0.97888	0.58523	0.92808	0.97744
0.25	16	0.58239	0.92842	0.97741	0.58400	0.92820	0.97751
	D	0.58390	0.92821	0.97751	0.58390	0.92821	0.97751
	4	0.43368	0.77501	0.90173	0.41669	0.75274	0.90296
	8	0.41034	0.75963	0.90424	0.41986	0.75963	0.90329
1	16	0.41440	0.75951	0.90249	0.41848	0.75952	0.90328
	D	0.41852	0.75952	0.90328	0.41852	0.75952	0.90328
	4	0.26278	0.47880	0.65036	0.27140	0.47881	0.65115
	8	0.26124	0.47953	0.65203	0.26807	0.48081	0.65177
4	16	0.26469	0.48053	0.65077	0.26925	0.48081	0.65180
	D	0.26746	0.48069	0.65178	0.26746	0.48069	0.65178
4	4	0.11592	0.21118	0.28993	0.12145	0.21384	0.29282
	8	0.11604	0.21260	0.29254	0.12594	0.21431	0.29305
16	16	0.11770	0.21329	0.29225	0.12406	0.21400	0.29293
	D	0.11897	0.21342	0.29279	0.11897	0.21342	0.29279

"zero to within machine accuracy." We have found this to be a very practical and useful approach, and as can be judged from Tables 3 and 4, it produces excellent results.

b. Comparison with other established methods

The most ambitious attempt to compare existing methods for the solution of radiative transfer problems in scattering atmospheres was initiated by the Radiation Commission of the International Association of Meteorology and Atmospheric Physics. The results of this intercomparison of different computational procedures were recently published in a report edited by J. Lenoble (cf. Lenoble, 1977; hereafter referred to as "Standard Procedures").

In Tables 5 and 6 we compare the present results for reflected and transmitted fluxes with those obtained by a variety of other methods given in "Standard Procedures." The physical situation is as follows: A parallel beam of radiation is incident on a plane parallel layer of haze particles (Haze L, cf. Deirmendjian, 1969) of total optical thickness $\tau^* = 1$. The scattering phase function and its coefficients in a Legendre polynomial expansion are tabulated in "Standard Procedures." The vertical flux of the incident solar beam in direction θ_0 = arc $\cos\mu_0$ was assumed to be $\mu_0 I^-(0) = \mu_0$ $(I^-(0) = 1)$.

In Table 5 we show comparisons for single scattering albedo $\omega = 0.9$ and for an overhead sun ($\mu_0 = -1.0$) as well as for a solar zenith angle of 60° ($\mu_0 = -0.5$). We notice that the spherical harmonic, the matrix operator and the doubling methods yield essentially identical results. These three methods are the most accurate ones considered in

Table 5

Comparison of reflected flux $[F^+(0)]$, diffuse transmitted flux $[F^-(\tau^*)]$, and net flux at the top $[F(0)]$ and bottom $[F(\tau^*)]$ of the atmosphere computed by various methods for single scattering albedo $\omega = 0.9$

METHOD	$F^+(0)$	$F^-(\tau^*)$	$F(0)$	$F(\tau^*)$
	$\mu_0 = -1.0$			
Present Results Number of Streams				
4	0.1207	1.5274	3.0209	2.6831
8	0.1238	1.5159	3.0178	2.6716
16	0.1237	1.5155	3.0179	2.6712
SPCART*)				
Spherical harmonics	0.1236	1.5155	3.0180	2.6712
Matrix operator	0.1237	1.5156	3.0179	2.6713
Monte Carlo	0.1230	1.516	3.019	2.672
Discrete ordinates	0.1237	1.5155	3.0178	2.6714
Doubling	0.1237	1.5155	3.0179	2.6713
	$\mu_0 = -0.5$			
Present Results Number of Streams				
4	0.2371	0.7948	1.3337	1.0074
8	0.2251	0.8033	1.3457	1.0159
16	0.2255	0.8033	1.3453	1.0159
SPCART*)				
Spherical harmonics	0.2255	0.8032	1.3452	1.0158
Successive scattering	0.2257	0.8036	1.3453	1.0162
Monte Carlo	0.234	0.795	1.336	1.008
Discrete ordinates	0.2262	0.8035	1.3446	1.0161
Doubling	0.2255	0.8033	1.3453	1.0159

*) See "Standard Procedures to Compute Atmospheric Radiative Transfer in a Scattering Atmosphere," J. Lenoble, ed. (1977).

Table 6

Comparison of reflected flux [$F^+(0)$], diffuse transmitted flux [$F^-(\tau^*)$], and net flux at the top [$F(0)$] and bottom [$F(\tau^*)$] of the atmosphere computed by various methods for conservative scattering ($\omega = 1.0$)

$\mu_0 = -1.0$

Present Results Number of Streams	$F^+(0)$	$F^-(\tau^*)$	$F(0)$	$F(\tau^*)$
4	0.1634	1.8225	2.9782	2.9782
8	0.1733	1.8126	2.9683	2.9683
16	0.1733	1.8126	2.9683	2.9683
SPCART*)				
Spherical harmonic	0.1736	1.8124	2.9680	2.9682
Successive scattering	0.1734	1.7954	2.9688	2.9512
Monte Carlo	0.165	1.820	2.976	2.976
Discrete ordinates	0.1732	1.8127	2.9644	2.9684
Doubling	0.1732	1.8126	2.9684	2.9684

*) See "Standard Procedures to Compute Atmospheric Radiative Transfer in a Scattering Atmosphere," J. Lenoble, ed. (1977).

"Standard Procedures" (the reader is referred to this report and references therein for further information pertaining to these methods). The present 16-stream results are in excellent agreement with those obtained by the former methods. We further note that our 8-stream results yield an accuracy of the order of one tenth of a percent or better while our 4-stream approximation is accurate to within a few percent.

Table 6 shows a similar comparison for the same phase function but for conservative scattering ($\omega = 1$) and for an overhead sun ($\mu_0 = -1.0$) only. We have treated this case of perfect scattering as discussed above and again we find that there is excellent agreement with the most accurate of the results given in "Standard Procedures."

c. Rayleigh scattering--comparison with Dave and Canosa (1974)

Following Dave and Canosa we adopt the phase function $p_{RAY}(\cos\Theta) = 3/4(1 + \cos^2\Theta)$ to represent Rayleigh scattering. The $\cos^2\Theta$-dependence implies that this phase function has an asymmetry factor, $g = 0$, and that only the $\ell = 0$ and the $\ell = 2$ terms in a Legendre polynomial expansion contribute.

Dave and Canosa (1974) give values of diffuse and net fluxes at the top and bottom of a perfectly scattering Rayleigh atmosphere of optical thickness $\tau^* = 1$ for an overhead sun ($\Theta = 0°$) and for a solar zenith angle of 80°. The vertical incident flux at the top of the atmosphere was taken to be μ_0 ($I^-(0) = 1$). In Table 7 we compare results given by Dave and Canosa (1974) using a spherical harmonic method (Sp.H.) keeping up to 100 terms in the expansion with the present discrete ordinate results (DOM) using 4, 8, and 16 streams. It is seen that the agreement is very good.

Table 7

Comparison of diffuse and net fluxes at the top and bottom of a non-absorbing Rayleigh atmosphere computed by a spherical harmonics method (Dave and Canosa, 1974) and by discrete ordinates (present results)

		DIFFUSE FLUX				NET FLUX				ABSORPTIVITY	
		at top $F^+(0)$		at bottom $F^-(\tau^*)$		at top $F(0)$		at bottom $F(\tau^*)$		$F^+(0)-F^-(\tau^*)$	
Sp. H. Terms	DOM Streams	Sp.H.	DOM	Sp.H.	DOM	Sp.H.	DOM	Sp.H.	DOM	Sp.H.	DOM
$\Theta = 0°$											
6	4	1.0704	1.0715	0.9156	0.9144	2.0712	2.0701	2.0713	2.0701	-0.0001	0.0000
20	8	1.0699	1.0697	0.9160	0.9162	2.0717	2.0719	2.0717	2.0719	0.0000	0.0000
60	16	1.0703	1.0696	0.9156	0.9163	2.0712	2.0720	2.0713	2.0720	-0.0001	0.0000
100		1.0706		0.9151		2.0710		2.0708		0.0002	
$\Theta = 80°$											
6	4	0.3627	0.3651	0.1813	0.1804	0.1828	0.1804	0.1831	0.1804	-0.0003	0.0000
20	8	0.3622	0.3618	0.1819	0.1820	0.1833	0.1837	0.1836	0.1837	-0.0003	0.0000
60	16	0.3622	0.3618	0.1819	0.1820	0.1834	0.1837	0.1836	0.1837	-0.0002	0.0000
100		0.3623		0.1818		0.1833		0.1835		-0.0002	

We notice, however, that while Dave and Canosa have difficulties with spurious absorption, this is never a problem in the present approach. As shown by Wiscombe (1977) this is due to the fact that there is no need for phase function renormalization in the discrete ordinate method when Gaussian quadrature is used. As a result, flux conservation (to within machine accuracy) is guaranteed for conservative scattering.

d. Comparison with Wiscombe (1977)

It was assumed in Section 2 that the phase function can be adequately approximated by a finite sum of Legendre polynomials. This representation is particularly convenient since it has been proven that it is the only one that allows an expansion of the intensity in a Fourier series in azimuth (Shifrin et al., 1972; see Lenoble, 1977). The latter expansion is useful because it leads to a set of mathematically similar but independent equations, one for each Fourier component, so that whatever techniques are available for solution of the zeroth order component can also be used for all higher order components.

The only problem with this approach is that if the phase function is strongly asymmetric, it cannot be accurately approximated by polynomials of low degree. This is the case for phase functions typical of atmospheric haze and cloud particles which are strongly peaked in the forward direction.

In the n^{th} discrete ordinate approximation the number of terms to be used in the phase function expansion is 2n. Thus, it is seen that if n is chosen to be large so as to yield an accurate phase function representation, a high-order and thus costly discrete ordinate solution is required. Moreover, although in principle the accuracy of the discrete ordinate solution will increase with increasing order, in practice this may not be so because the matrix manipulations required to determine the eigensolutions become more susceptible to computer round-off errors when the order of the system is increased. These problems are discussed in a recent paper by Wiscombe (1977) who introduced a new phase function representation for efficiently computing fluxes in scattering media with strongly forward peaked phase functions. We have adopted Wiscombe's approach here and for the purpose of the following comparison a brief summary of this technique is given below [for a more comprehensive account see Wiscombe (1977)]. In the following we compare the result of using this new representation with that of using a regular Legendre polynomial expansion of the phase function.

The fact that the phase function is strongly peaked in the forward direction suggests trying an expansion of the form (Joseph et al., 1976; Wiscombe, 1977)

$$p_{\delta-L}(\cos\Theta) = 2f\,\delta(1-\cos\Theta) + (1-f)\sum_{\ell=0}^{2n-1}(2\ell+1)g_\ell\,P_\ell(\cos\Theta) \tag{88}$$

where P_ℓ $(\cos\Theta)$ is the Legendre polynomial. The subscript "δ-L" on $p(\cos\Theta)$ is a reminder that this phase function consists of a superposition of a delta-function (δ) and an expansion in Legendre polynomials (L). f denotes the strength of the peak and, clearly, if $f = 0$ (88) is just a regular expansion in Legendre polynomials. The coefficients g and f are determined as follows.

Defining

$$\chi_m \equiv \int_{4\pi} P_m(\cos\,\Theta)\; p_{ac}(\cos\,\Theta)\; \frac{d\Omega}{4\pi} \qquad ; \; m = 1,2,\ldots,2n \tag{89}$$

where p_{ac} $(\cos\,\Theta)$ is the actual phase function, the g_ℓ's are determined by requiring

$$\int_{4\pi} P_m(\cos\,\Theta)\; p_{\delta-L}(\cos\,\Theta)\; \frac{d\Omega}{4\pi} \equiv \chi_m \tag{90}$$

which leads to

$$g_\ell = \frac{\chi_\ell - f}{1 - f} \qquad ; \; \ell = 1,2,\ldots,2n\text{-}1 \tag{91}$$

$$f \equiv \chi_{2n} \qquad \text{(truncation)} \tag{92}$$

We note that although the truncation is arbitrary it has the desirable feature that f depends on the order of the approximation such that $f \to 0$ as $n \to \infty$. Furthermore, it is easily checked that the representation (88) has the correct normalization since

$$\int_{4\pi} p_{\delta-L}(\cos\,\Theta)\; \frac{d\Omega}{4\pi} = 1 \tag{93}$$

Another attractive feature of this representation is that when (88) is substituted into the transfer equation a new equation, which is identical in form to the original one, is obtained. The only difference is that the single scattering albedo, ω', and the optical depth, τ', of the new equation are related to those of the original one (unprimed) through

$$\omega' = \frac{(1-f)\omega}{1-\omega f} \quad (94)$$

$$d\tau' = (1-\omega f)d\tau \quad (95)$$

Since the form of the transfer equation has not changed, it follows that whatever techniques are available to solve the original equation can be used to solve the new equation. This holds true in particular for the discrete ordinate method. In fact one may argue that the discrete ordinate solution to the new equation represents the most logical and direct extension of the delta-Eddington approximation of Joseph et al. (1976).

In Table 8 we show plane albedo values for an atmosphere of unit optical thickness for a single scattering albedo $\omega = 0.8$ and for a Henyey-Greenstein phase function with various asymmetry factors, g. Discrete ordinate results using a maximum of 64 streams are shown for a regular Legendre polynomial expansion of the phase function denoted by DOM (discrete ordinate method) and for Wiscombe's δ-L representation denoted by δ-DOM.

We note that even for $g = 0.75$ which is not a very asymmetric phase function the results for δ-DOM are slightly better and converge more rapidly with increasing number of streams than those for DOM. This tendency is drastically accentuated for larger g-values and it is seen that for $g \gtrsim 0.85$ it is mandatory to use δ-DOM if accurate flux values are to be obtained in low order (less than 16 streams) approximations.

Table 8

Plane albedo for an atmosphere of unit optical thickness with single scattering albedo $\omega = 0.8$ and for a Henyey-Greenstein phase function with various asymmetry factors, g

	DOM			δ-DOM		
		μ_0			μ_0	
Number of streams	0.1	0.5	0.9	0.1	0.5	0.9
			g = 0.75			
4	0.38393	0.12637	0.05099	0.34599	0.12880	0.04964
8	0.35815	0.12310	0.04856	0.35367	0.12313	0.04923
16	0.35491	0.12340	0.04929	0.35495	0.12342	0.04929
32	0.35487	0.12342	0.04929	0.35487	0.12342	0.04929
			g = 0.85			
4	0.36252	0.08070	0.02871	0.28499	0.08429	0.02662
8	0.31683	0.07665	0.02396	0.30254	0.07701	0.02639
16	0.30644	0.07738	0.02665	0.30664	0.07760	0.02653
32	0.30598	0.07759	0.02652	0.30602	0.07759	0.02652
			g = 0.95			
4	0.33370	0.02376	0.01051	0.15457	0.03011	0.00724
8	0.23362	0.02036	-0.00296	0.18146	0.02344	0.00716
16	0.19000	0.02250	0.00870	0.18928	0.02449	0.00743
32	0.18350	0.02450	0.00745	0.18569	0.02444	0.00740
64	0.18563	0.02444	0.00740	0.18555	0.02445	0.00740

Thus it is seen that incorporating the phase function representation invented by Wiscombe (1977) into the present scheme results in a powerful tool for solving transfer problems involving strongly forward peaked scattering.

e. Spherical albedo for a semi-infinite atmosphere—comparison with Dlugach and Yanovitskij (1974)

In order to further assess the usefulness of low orders of approximation spherical albedo values were computed for single scattering albedo, ω, ranging from 0.5 to 0.99999 and asymmetry factor, g, ranging from 0.3 to 0.99. A Henyey-Greenstein phase function

$$p(\cos\Theta) = \frac{1-g^2}{(1+g^2-2g\cos\Theta)^{3/2}} \tag{96}$$

with asymmetry factor, g, was utilized. The incident intensity was taken to be isotropic over the downward hemisphere. In these computations Wiscombe's (1977) δ-L representation for the phase function was used. The results are shown in Table 9. The values for the spherical albedo, A_s, in this table are calculated directly from the discrete ordinate method using the Gaussian quadrature, i.e.

$$\begin{aligned} A_s \equiv \frac{F^+(\tau=0)}{F^-(\tau-0)} &= \frac{2}{I^-(0)} \sum_{i=1}^{n} a_i\mu_i \, I_i(\tau=0) \\ &= \frac{2}{I^-(0)} \sum_{i=1}^{n} a_i\mu_i \sum_{j=1}^{n} L_j g_j(\mu_i) \end{aligned} \tag{97}$$

where $F^+(\tau = 0)$ is the reflected flux for an isotropic incident illumination (which is proportional to the spherical albedo for a parallel incident

Table 9

Flux albedos in different orders of approximation
for a wide range of single scattering albedos
and asymmetry factors

Appr.	g =.3	.5	.7	.9	.95	.99	
2	0.1318	0.1010	0.0655	0.0238	0.0122	0.0025	
	0.1143	0.0846	0.0524	0.0177	0.0089	0.0018	
4	0.1115	0.0845	0.0544	0.0199	0.0103	0.0021	
6	0.1099	0.0826	0.0524	0.0185	0.0094	0.0019	ω = .5
8	0.1098	0.0825	0.0522	0.0183	0.0092	0.0019	
16	0.1098	0.0824	0.0521	0.0183	0.0092	0.0018	
2	0.2374	0.1909	0.1319	0.0524	0.0276	0.0058	
	0.2100	0.1644	0.1093	0.0402	0.0204	0.0041	
4	0.2060	0.1635	0.1113	0.0438	0.0232	0.0049	
6	0.2043	0.1615	0.1088	0.0414	0.0214	0.0044	ω = .7
8	0.2042	0.1613	0.1087	0.0412	0.0212	0.0043	
16	0.2042	0.1613	0.1086	0.0412	0.0212	0.0043	
2	0.4407	0.4021	0.3159	0.1591	0.0926	0.0205	
	0.4222	0.3641	0.2796	0.1318	0.0731	0.0157	
4	0.4183	0.3613	0.2784	0.1352	0.0778	0.0182	
6	0.4174	0.3602	0.2769	0.1326	0.0749	0.0166	ω = .9
8	0.4173	0.3602	0.2769	0.1326	0.0748	0.0165	
16	0.4173	0.3602	0.2768	0.1326	0.0748	0.0165	
2	0.5817	0.5283	0.4427	0.2601	0.1654	0.0435	
	0.5446	0.4899	0.4040	0.2256	0.1372	0.0325	
4	0.5418	0.4870	0.4009	0.2261	0.1408	0.0365	
6	0.5414	0.4864	0.4002	0.2244	0.1382	0.0341	ω = .95
8	0.5413	0.4864	0.4001	0.2244	0.1382	0.0340	
16	0.5413	0.4864	0.4001	0.2244	0.1383	0.0340	
2	0.9925	0.9910	0.9885	0.9802	0.9721	0.9387	
	0.9913	0.9898	0.9869	0.9778	0.9688	0.9318	
4	0.9913	0.9897	0.9868	0.9772	0.9678	0.9296	
6	0.9913	0.9897	0.9868	0.9772	0.9679	0.9298	ω = .99999
8	0.9913	0.9897	0.9867	0.9772	0.9679	0.9298	
16	0.9913	0.9897	0.9867	0.9772	0.9679	0.9298	

beam). Note that the exponentially growing solutions are not used in this case of a semi-infinite atmosphere. For the two-stream approximation the improved albedo value resulting from using the method of Swartz and Stamnes (1977) is included in Table 9. This is the more accurate value given for the two-stream approximation.

Table 9 demonstrates the following assertions:

1) Even the 2-stream approximation gives adequate results when the improved albedo is used.

2) The accuracy of 2- and 4-stream approximations does not deteriorate with increasing asymmetry.

3) Six-stream solutions give very accurate results and from a practical point of view the method has essentially converged for 8 streams.

Spherical albedo values for a semi-infinite scattering atmosphere have been previously published by Dlugach and Yanovitskij (1974) who used an accurate iteration technique to solve the equation of transfer. Their results (given to 3 decimal places) are in complete agreement with ours.

9. CONCLUSION

The difficulties with Chandrasekhar's prescription for numerical implementation of the discrete ordinate method in radiative transfer as previously practiced are discussed. In order to overcome these problems a matrix formulation is provided so that the eigensolutions can be obtained by standard, reliable and efficient routines, specifically designed to handle such problems. In particular it is shown that the order of the algebraic eigenvalue problem can be reduced by a factor of 2.

A new expression for the source function is derived and used to obtain expressions for the angular distributions at angles other than the quadrature angles. By appealing to the reciprocity principle it is shown that certain quantities have a dual interpretation, and it is pointed out that this duality can be used in the discrete ordinate method to obtain very simple expressions for albedo and transmissivity. It is believed that this duality has been utilized in the Doubling method (cf. Irvine, 1975), but to this author's knowledge this is not the case for other methods commonly used to study the transfer of radiation in planetary atmospheres (cf. Irvine, 1975; Hansen and Travis, 1974; Hunt, 1971).

Since the discrete ordinate method and the spherical harmonic method are equivalent in the azimuth independent case (cf. e.g. Krook, 1955; Kofink, 1967), the instability problems experienced in the spherical harmonic method (Devaux and Herman, 1971; Devaux et al., 1973; Irvine, 1975) are certainly of the same nature as those encountered in the discrete ordinate method. Thus, it may well be that the problems of solving the characteristic equation in the spherical harmonic method could be overcome by properly formulating it as a standard algebraic eigenvalue problem.

Numerical verification of the theory is provided by comparing fluxes calculated by the present method with those of other established procedures. The high speed and accuracy of the present approach makes it a valuable tool for repetitive computations such as those encountered in atmospheric radiative budget calculations for which solutions of the radiative transfer equation are required for a large number of wavelengths in the solar and terrestrial spectrum and for a variety of atmospheric conditions (e.g., Braslau and Dave, 1973a, b; Yamamoto et al., 1971). Our approach can be applied to a multitude of problems involving anisotropic scattering of radiation in planetary atmospheres (cf. e.g., Chamberlain, 1976; Price, 1977; Toon et al., 1977).

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