

Highly incident configurations and reduced Levi graphs

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What is a configuration?

An (n,k) -configuration is a collection of points and lines in the Euclidean plane where each point lies on n lines and each line passes through k points.

If $n=k$, we call it a k -configuration.

Figure 1 is a 3-configuration; each line passes through 3 points and each point lies on 3 lines. The configuration in Figure 1 has 3-fold rotational symmetry which allows us to construct its reduced Levi graph.

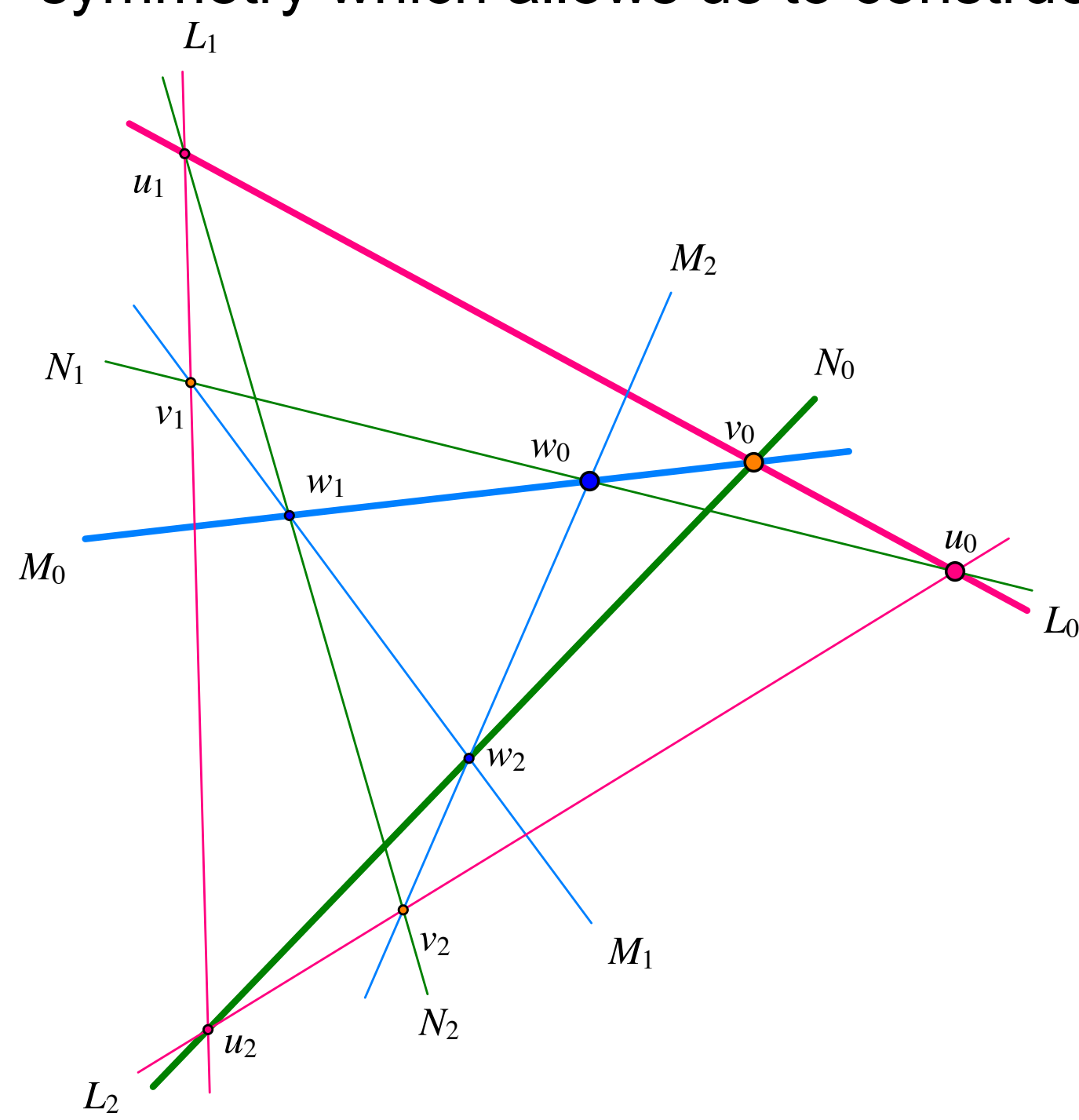


Figure 1: A (9_3) -configuration with 3-fold rotational symmetry.

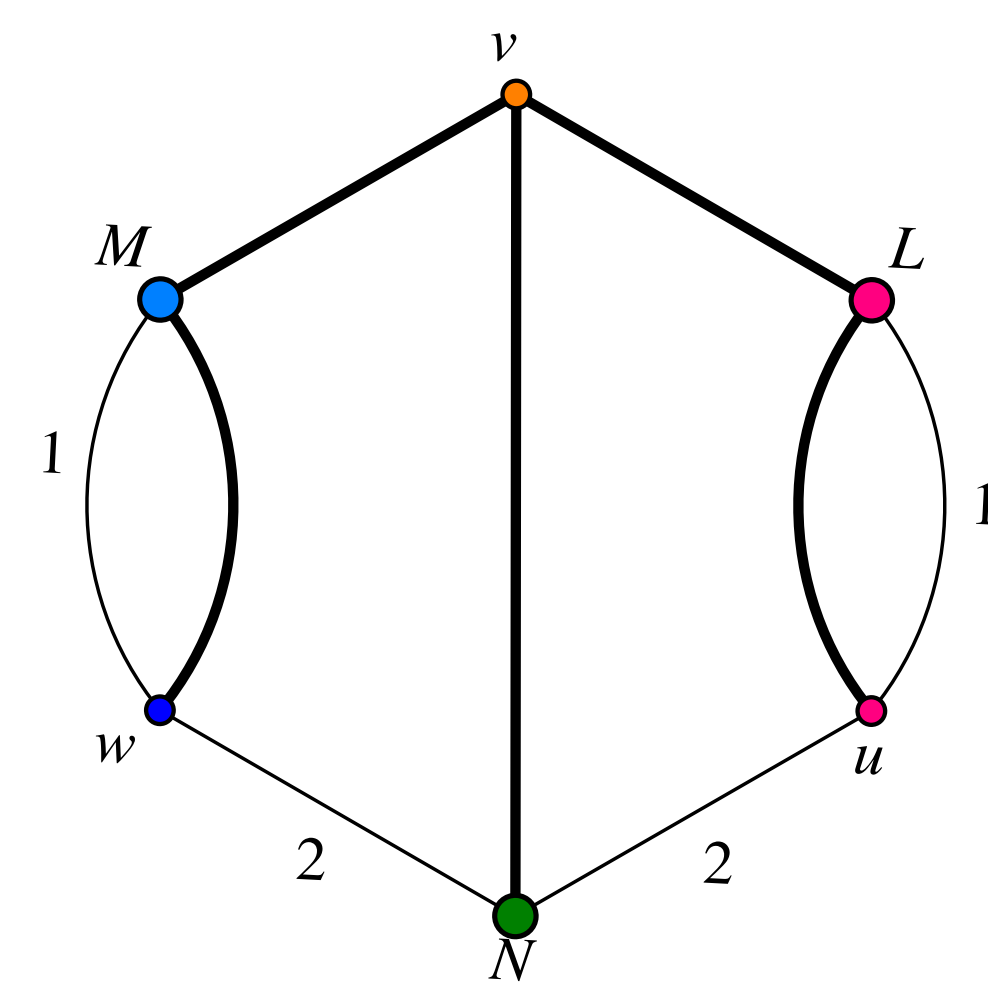


Figure 2: The reduced Levi graph of the configuration in Figure 1. The labels indicate which lines are incident with which points.

Figure 2 is the reduced Levi graph of the configuration in Figure 1. The reduced Levi graph of a configuration is an edge-labeled multi-graph that encodes geometric properties of the configuration.

In Figure 1, the line L_0 passes through points u_0 , u_1 , and v_0 . So, in Figure 2, the two edges between L and u are labeled 0 and 2 and the edge between L and v is bold to indicate a label of 0.

In general, line L_i passes through the point u_i and u_{i+1} and L_i passes through v_i where addition is mod 3.

Objectives

To find reduced Levi graphs of configurations, we attempted to label embedded cycles of the three graphs shown in Figure 3. We labeled these embedded cycles to be the reduced Levi graphs of a family of known configurations called trivial configurations.

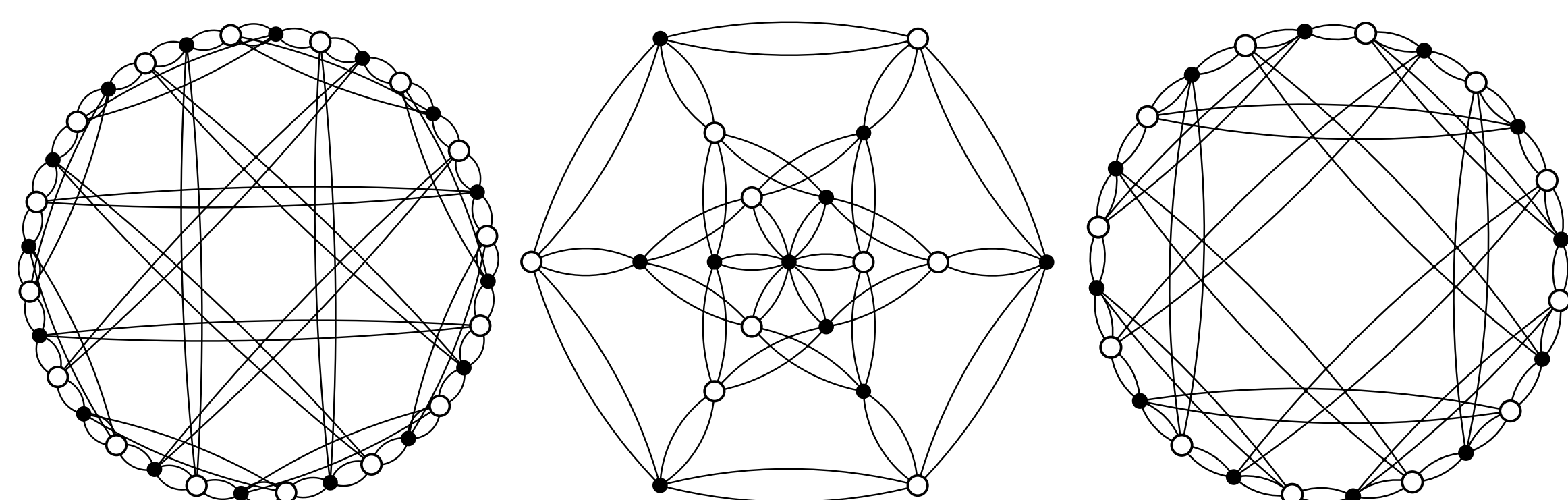


Figure 3: From left to right: the Dyck graph, the Pappus graph, and the Nauru graph each with every edge doubled.

Methods

To find a reduced Levi graph of a larger configuration, we did the following to each graph in Figure 3.

1. Label one of each double edge with a 0.
- 2a. Label a 6-cycle with (a,b,c,a,b,c) .
OR
- 2b. Label an 8-cycle with either (a,b,c,d,b,a,d,c) or (a,d,c,a,b,c,d,b) .
3. Force all other 6-cycles to be trivial, that is, of the form (a,b,c,a,b,c) or all other 8-cycles to be either (a,b,c,d,c,b,a,d,c) or (a,d,c,a,b,c,d,b) .
4. Check graphs for equal adjacent edges.

Results

This investigation did not result in any labelings that correspond to a reduced Levi graph. Here are a few of the results for each graph. Note the doubled edges are not shown for clarity.

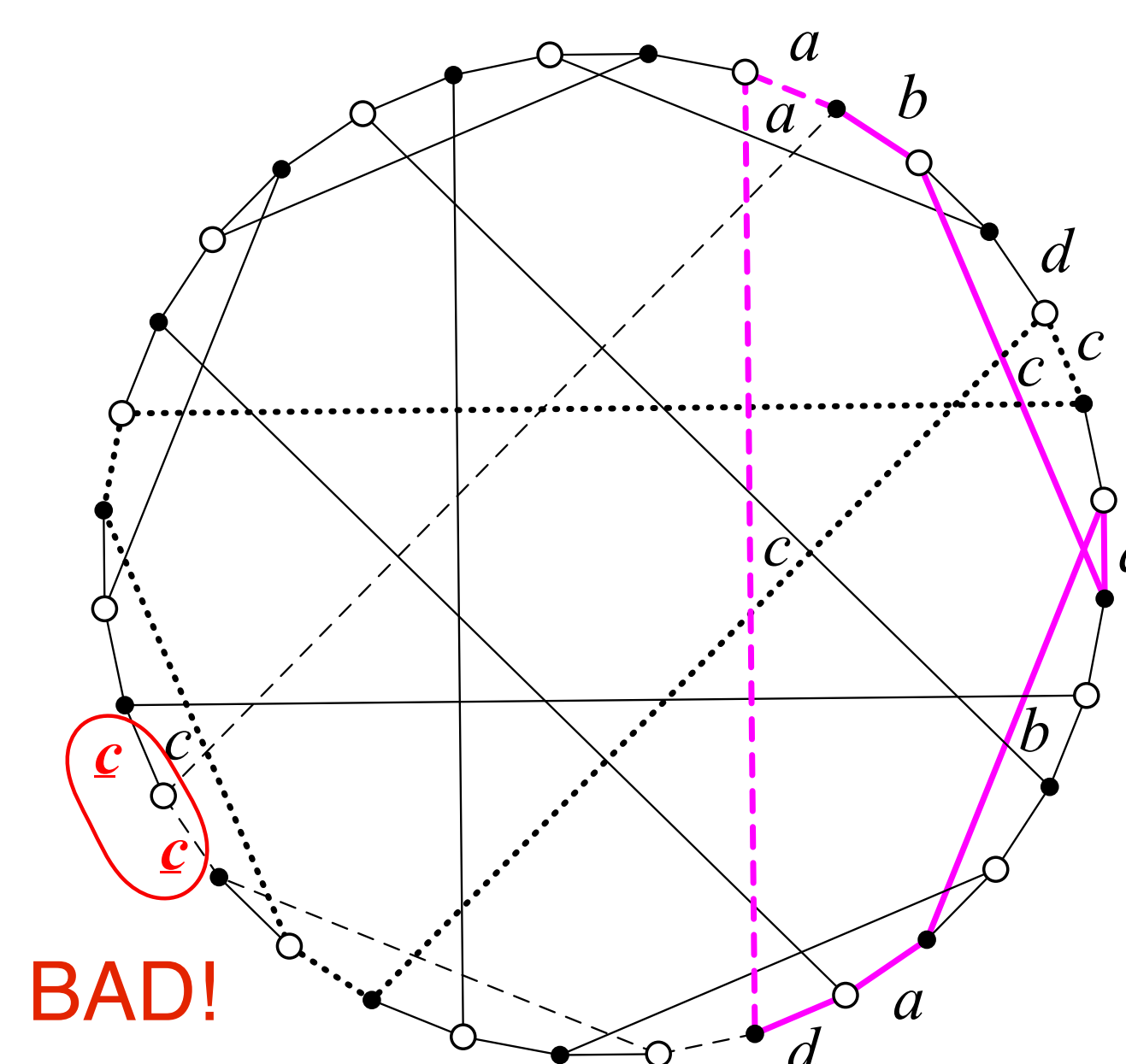


Figure 4: Attempted labeling of the Dyck graph with doubled edges.

Dyck Graph

We first labeled the pink 8-cycle (Figure 4). Forcing all trivial 6-cycles results in adjacent edges both labeled "c". A reduced Levi graph cannot have equal adjacent edges, disqualifying this labeling.

Pappus Graph

We then tried to label the graph in Figure 5. We chose a 6-cycle and labeled the edges with (a,b,c,a,b,c) . Then, we forced all other 6-cycles to be of the form (a,b,c,a,b,c) . Once again, we get equal adjacent spans, circled in red. So, Figure 5 is not a reduced Levi graph.

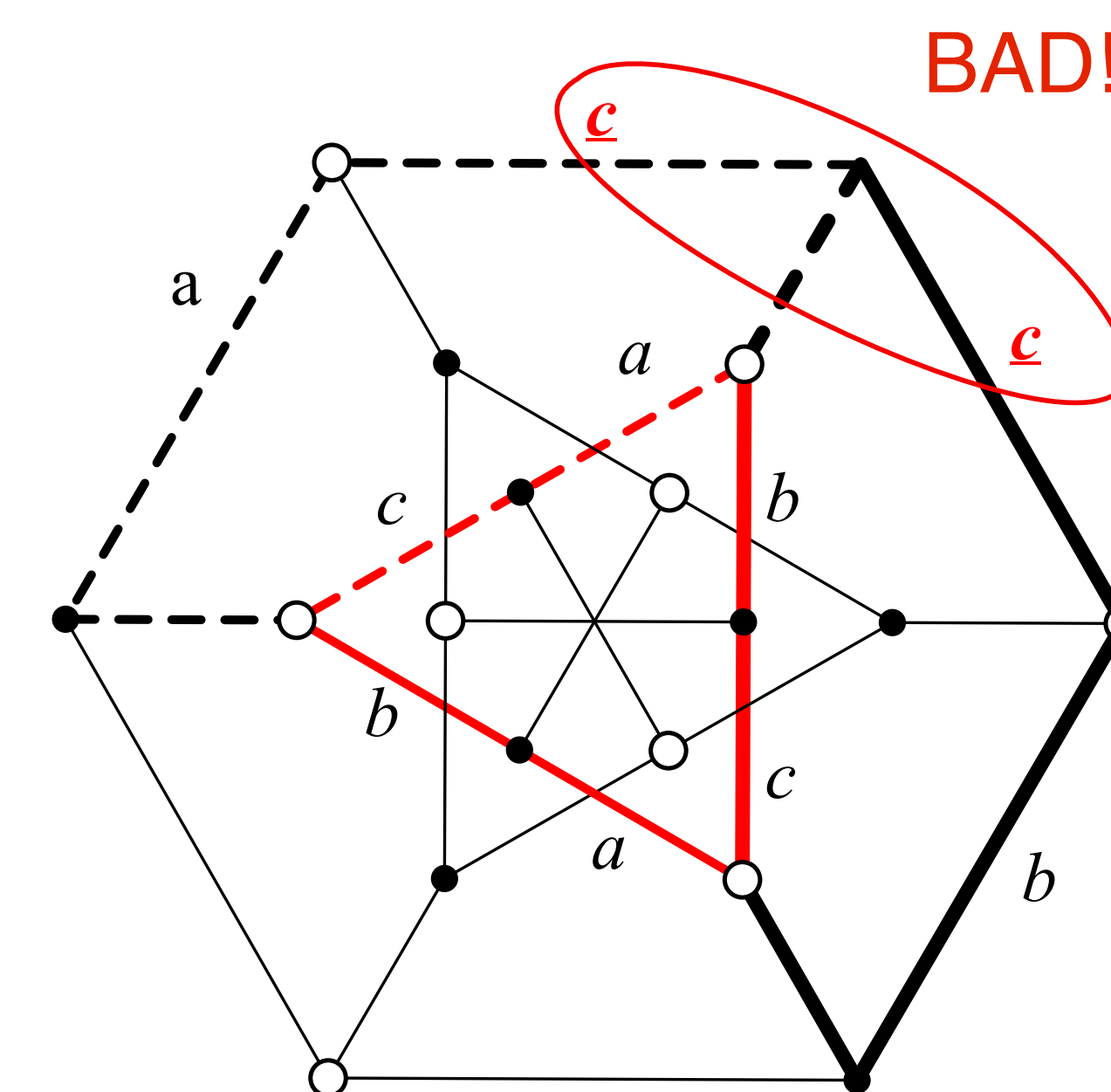


Figure 5: Attempted labeling of the Pappus graph with doubled edges.

Results continued

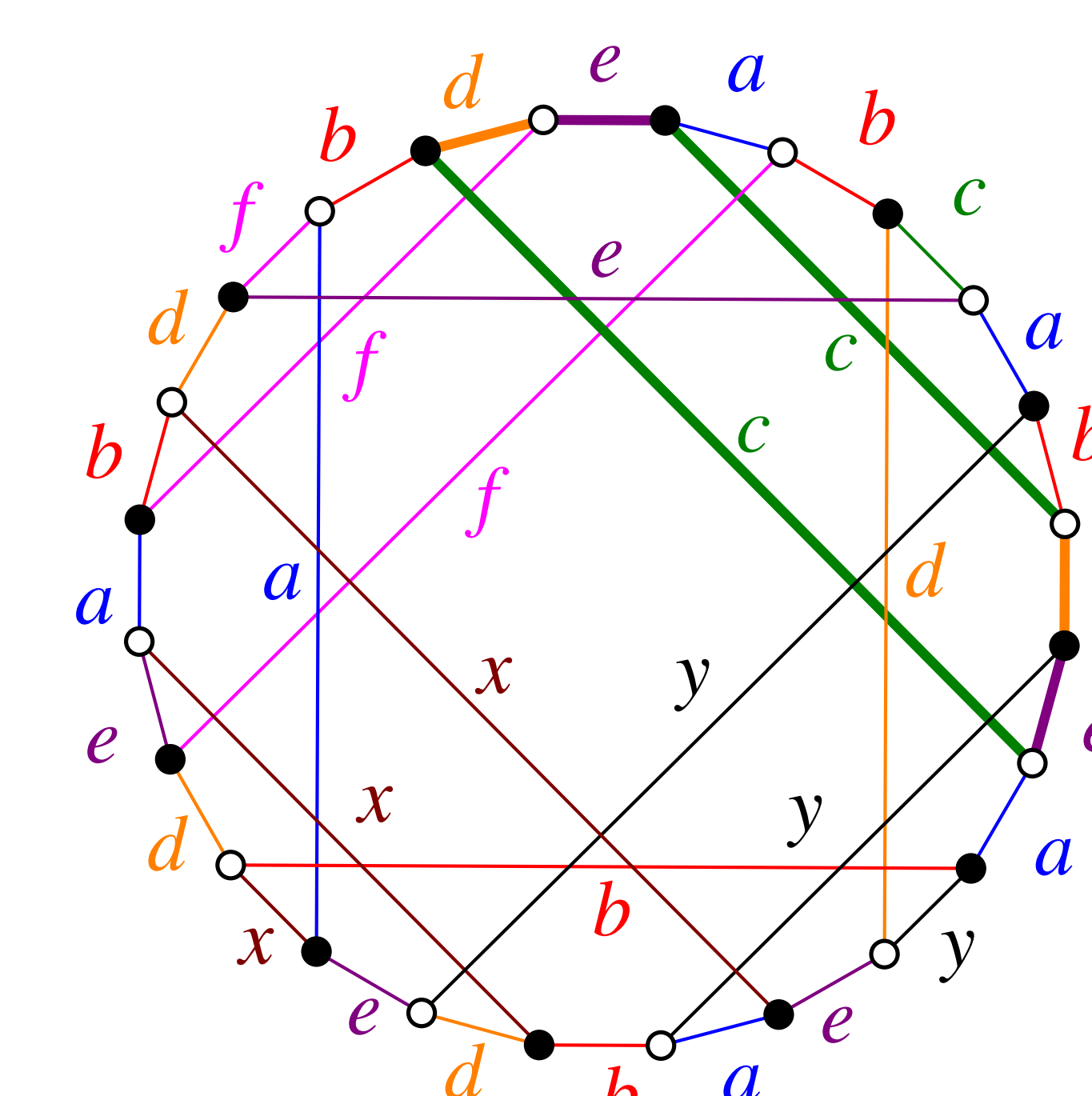


Figure 6: The Nauru Graph with doubled edges.

Nauru Graph

The Nauru graph with doubled edges produces a labeling with no equal adjacent edges. However, the 8-cycles are of the form (d,f,b,c,a,y,e,x) . This 8-cycle does not correspond to a trivial configuration.

(4,8)-configurations

During this investigation, we found three previously undiscovered (4,8)-configurations. Figure 7 gives one of them and its reduced Levi graph.

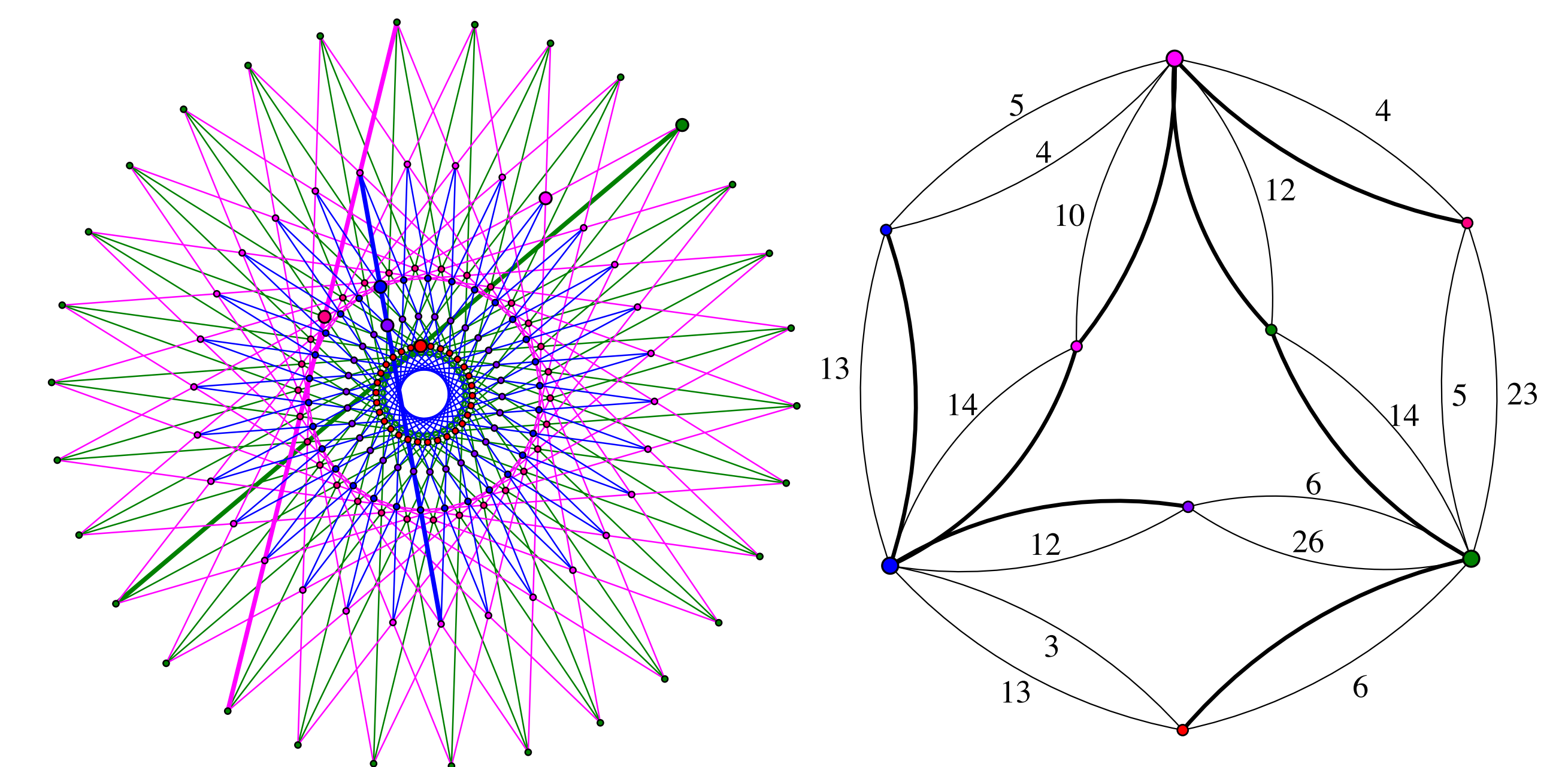


Figure 7: A (4,8)-configurations and its corresponding reduced Levi graph.

Open Questions

1. Are there choices of labels for the Nauru graph that result in the reduced Levi graph of a configuration?
2. What other graphs with each edge doubled might be a reduced Levi graph for a configuration?

Acknowledgements

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